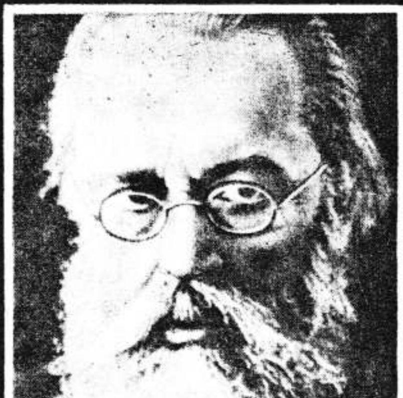
Academician STEKLOV Mathematician Extraordinary

Outstanding Soviet Scientists



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Academician STEKLOV
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V. S. VLADIMIROV and I. I. MARKUSH

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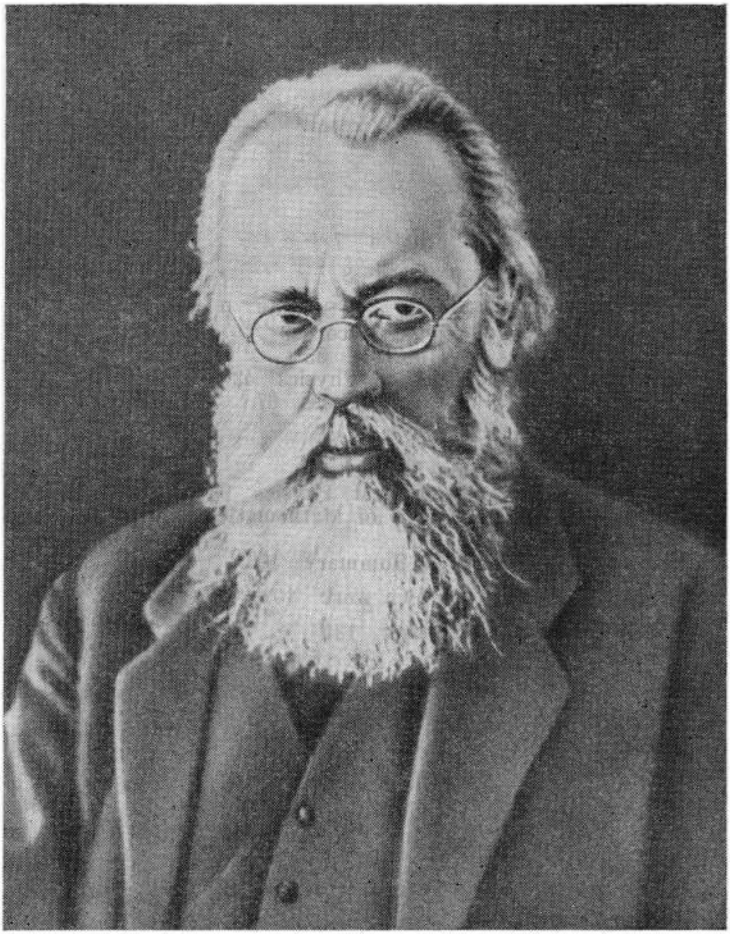

Vladimir Andreevich Steklov
(1864-1926)

PREFACE TO THE ENGLISH EDITION

In 1936 Academician A. N. Krylov wrote of V. A. Steklov that "...he should be numbered amongst that group of illustrious Russian mathematicians that includes Ostrogradskii, Chebyshev, and Lyapunov" [XIX]. It seemed obvious, therefore, to offer this essay on the life and activities of this remarkable Russian mathematician to overseas readers.

The English edition appears quite some time after the Russian edition and thus has some additions. These are chiefly an expansion of Steklov's biography and the inclusion of new photographs.

14th June, 1982 V. VLADIMIROV
Moscow/Uzhgorod I. MARKUSH

PREFACE TO THE RUSSIAN EDITION

This essay is dedicated to the Fiftieth Anniversary of the founding of the V. A. Steklov Mathematical Institute of the USSR Academy of Sciences. It is a biography of and a description of the scientific, educational, and social activities of an eminent Russian mathematician and mechanician, the founder of the Petersburg School of mathematical physics, an organiser of the Physico-Mathematical Institute of the Russian Academy of Sciences, Academician V. A. Steklov (1864-1926). It was written using archival materials, analyses of his work, unpublished correspondence (both with Russian and foreign scientists), reminiscence of his contemporaries, works on V. A. Steklov, and other data. An analysis is given in the essay of Steklov's basic work and it is shown how it has influenced the subsequent development of mathematics. A very full list of Steklov's work is provided.

This essay is an expanded and revised presentation of the material in our brochure *Academician V. A. Steklov*, Znanie, Moscow, 1973, 64 pp. ("Mathematics and Cybernetics" Series, No. 5).

As the authors were preparing this material they were greatly aided by conversations with Academician I. M. Vinogradov and the late Academician V. I. Smirnov. A number of the manuscript's paragraphs were read by Academician S. M. Nikolskii, Professor V. P. Mikhailov, and Professor V. P. Korobeinikov, who all made a number of valuable comments. The authors render all these people their most profound gratitude.

The Authors

I

A POTTED BIOGRAPHY OF V. A. STEKLOV

Vladimir Andreevich Steklov was born on the 9th January 1864 (28th December 1863 by the old calendar) in Nizhni Novgorod (now Gorky)*. His father, Andrei Ivanovich, coming from the background of the rural clergy, was the rector and a teacher at the Nizhegorod** clerical seminary, and later in the last two years of his life at the Tavrichesko-Simferopol seminary.

More information about the early years of V. A. Steklov's father, Andrei Ivanovich, can be found in an article written by Prof. P. M. Nikiforov, a close friend of V. A. Steklov. He wrote, "Andrei Steklov graduated at the top of his class from the Nizhegorod clerical seminary and was given a state scholarship to the Kazan ecclesiastical academy. In addition to a genius for science, he shared with his brothers a fine and powerful voice. Andrei Ivanovich felt no inclination for clerical study and having been sent to the ecclesiastical academy made every effort to get transferred to a university. Having exhausted every legal avenue, he resorted to more devious means, writing, for example heretical essays that would of necessity, exclude him from the academy. To no avail, he was given penances to do but not released from the academy and so, straightened by circumstances and

* Nizhni Novgorod lies on the right bank of the Volga where it is joined by the river Oka. Beginning from the thirteenth century, it expanded into a beautiful and prosperous town, becoming a stronghold against the Tartar inroads into Eastern Russia. The distinguished inventor and mechanic I. Kulibin, the great geometer N. Lobachevskii, and the revolutionary-democrat Nikolai Dobrolyubov were among the eminent personae in the arts and sciences who were born there.

** An abbreviation of Nizhni Novgorod.—*Ed.*

driven by material need, Andrei Ivanovich was forced to accept a clerical title. When he graduated from the academy (in 1854) he obtained a post as a teacher of Russian history and Hebrew at the Nizhegorod clerical seminary" [VI].

V. A. Steklov's mother, Ekaterina Aleksandrovna, was a sister of the distinguished Russian critic and publicist, the revolutionary-democrat, N. A. Dobrolyubov.

E. A. Dobrolyubova was orphaned when she was eleven, together with four sisters and three brothers. The eldest child in the family, Nikolai, was 18 whilst the youngest sister, Elizabeta, had only just been born. It was her birth that caused their mother to die; she passed away in March 1854 and her husband in August of the same year.

In 1855 Katya (Ekaterina) Dobrolyubova was sent to the Simbirsk* ecclesiastical orphanage for girls, "maintained at public expense". Life in the different atmosphere of the monastic orphanage was joyless but she was patient. Then in August 1858 15-year-old Katya left the orphanage and travelled to Nizhni Novgorod. There she lived with her elder sister Antonina who had got married not long before. Three years later, at 18, she married Andrei Ivanovich Steklov.

The household of young Andrei and Ekaterina Steklov was ruled by the ideas of the progressive liberators of the 1860's, particularly those of N. G. Chernyshevskii and N. A. Dobrolyubov. These names were mentioned with respect, not only in the Steklov family, but also in the circle of their close Nizhegorod acquaintances.

In 1874 V. A. Steklov, after receiving a good grounding at home, joined the first year of the Nizhegorod Aleksandrov Institute, which ran a secondary school curriculum**. During his first five years at the Institute, Steklov showed no especial interest in studying. He was a lively, enterprising, and audacious boy, the leader in a gang of his contemporaries and an indispensable participant in boyish pranks and games. All this, as he later remembered, "promoted good health and a strong nervous system, but did nothing for success in passing courses" [XXXII,

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* Now Ulyanovsk.—*Ed.*

** At this time V. A. Steklov had three younger sisters: 7-year-old Vera, 5-year-old Nadezhda, and 3-year-old Zinaida. They all later went to a gymnasium.

op. 3, No. 195]. However, after he had finished the fifth year, he thoroughly revised the fifth year subjects during the holidays, and by the end of the first quarter of the sixth year he was one of the best students. Henceforth, Steklov not only studied the compulsory subjects with interest, but also expanded his knowledge comprehensively, and became interested in physics, chemistry, and mathematics. Together with his uncle, a priest in one of the churches in Nizhni Novgorod, he established a laboratory at home for his studies in physics and chemistry.

In 1882 he graduated from the Institute with the silver medal, having been awarded distinctions in all subjects. The gold medal was not awarded to him because the school board thought that he had started to show "instability in his views and the symptoms of a pernicious trend", that were particularly revealed in a set essay of his entitled "The Fair Age Was the Age of Catherine".

In the same year Steklov started on the first course at the Physico-Mathematical Faculty of Moscow University. The first year was a failure for him. He passed all the basic examinations with Goods and Distinctions, but obtained an unsatisfactory mark for Physical Geography from Prof. A. G. Stoletov. This failure was a heavy blow for Steklov's proud nature. He was used to being the foremost amongst his contemporaries, justly, though perhaps vaguely, aware of his own great talent and rare strength of character. After this setback he contemplated transferring to the Medical Faculty, but the place did not materialise and so he joined the first year at the Mathematics Faculty of Kharkov University. Here, as his fellow student N. V. Raevskii has noted, Steklov was the centre of a circle of students devoted to science. His first two courses in mathematics were taught by Prof. M. F. Kovalskii, who was able to instil into students a love and respect for science.

In 1885, a young assistant professor, the talented mathematician A. M. Lyapunov (1857-1918), moved to Kharkov University from St. Petersburg, having defended his master's thesis *On the Stability of Equilibrium Ellipsoidal Forms in Rotating Liquids* shortly before. He was a student of P. L. Chebyshev and later became a full member of the Academy of Sciences. Steklov soon became a close disciple of his, and this influenced, in essence, all Stek-

lov's later scientific work. An intimate bond was established between tutor and pupil that was perhaps closer because there was only a seven year age difference between them, and to the end of his days Steklov retained the deepest respect for his teacher. Much later, at a meeting in 1919 of the Academy of Sciences given in his memory, Steklov spoke of Lyapunov: "A. M. could, even without being aware of it himself, subdue a hostile audience within a single hour by the force of his talent, a quality most young people respect unconditionally" [VIII].

In 1887, Steklov graduated from Kharkov University and remained to work towards a professorship. During the summer of 1888 Steklov went to the estates of a friend, A. S. Postnikov, who had been a professor at Odessa University. Here he did some interesting work of his own on the movement of rigid bodies in liquids. However, during a conversation with his tutor Lyapunov, Steklov found out that the integration solution of the differential equations of motion that he had come across had already been discovered by R. F. A. Clebsch. Despite this, Steklov persisted in his investigations and soon made another discovery that was actually more important. Lyapunov compared it with the discovery by S. V. Kovalevskaya of a new integration solution of the equations of motion for a rigid body around a stationary point.

At the same time Steklov was busy with other independent studies. As a result, in 1889 his first work appeared in print [1]. Henceforward, and throughout his life, his scientific work was unfailing and habitually frequent.

Concurrent with these scientific pursuits, Steklov was also preparing for his master's examination. He passed this successfully in 1890 and got married in the same year, his bride being Olga Nikolaevna Drakina.

Steklov and O. N. Drakina had met when he was an undergraduate, as her brother, Nikolai Drakin was a friend of his. Once they had met, Vladimir Andreevich became a frequent visitor to the Drakin family. He later remembered, "Olga would play the piano accompanying me and I would spend the whole evenings practising my singing" [XXXII, op. 3, No. 195]. Olga's father was a merchant although he had come from a peasant background in Orlov gubernia. He had walked to Kharkov as a

boy and at first sold matches on a corner. He had, however, managed to raise himself out of poverty, establish a business, marry and acquire his own house. Olga's mother was an uneducated woman with a despotic character. Olga, an intelligent and sensitive girl, was not happy with the lower middle class atmosphere of the household and her mother often reproached her for she was dissatisfied that her daughter could not find herself a fiancé. Although she had the support of her brother Nikolai at home and was close to her younger sister, Elena, her life at home was difficult and tense.

V. A. Steklov later wrote, "My love for Olga was not like my earlier ones, it came quietly, slowly, and grew like a sort of silent emotion devoid of rapid excesses. It was an utter and a tender sort of calm not at all like the sweeping gusts of emotion that possessed me during my relationships with Sonya*" [XXXII, op. 3, No. 195]. He was attracted to her quiet strength and independent mindedness. A lucid wisdom flowed from her, a quiet clean spirit, so it was not for nothing that those close to her called her an angel. Vladimir Andreevich later wrote, "Olga gradually became very dear to me and essential for my happiness. I did not know whether she loved me or not for she gave me few occasions to guess it. However, it seemed to me for a long time that I could spend my life with just such a person and I never wavered from this view. It is remarkable that she thought it impossible that I could ever love her even though it turned out that she had loved me for a long time. Later she told me she would have been satisfied even with an unrequited love for me, and been content for me to continue to visit her as a good friend like I had done before my proposal to her. When I was sure of my feelings for her I could do nothing but tell her of my love and ask her if she would agree to be my wife" [XXXII, op. 3, No. 195].

In 1891 Steklov was given a lecturing position as an assistant professor at Kharkov University. He began by lecturing on the theory of elasticity, which was a topic

* Sonya was V. A. Steklov's second cousin. They had had a child's love for each other. When Volodya (Vladimir) was 19 they found their love had not disappeared but had blazed up anew. However, Sonya eventually got married to a young official though she died in childbirth soon after.

he was most concerned with at that time. It should be noted that Steklov's interests from the beginning of his scientific career were very varied. Besides the research on elasticity theory [4, 5, 7, 28], he had already begun to do some work on hydrodynamics [2, 3, 8-12, 15, 17, 52] and higher algebra [1, 6], during the Kharkov period. Using results from several of these studies he successfully defended his thesis *On the Motion of a Rigid Body in a Liquid* [9] in 1893, and was awarded a Master's degree in applied mathematics in 1894.

In his review of this dissertation Lyapunov wrote, "...The whole paper was divided into five chapters. The first contained the derivation of the differential equations used to describe the motion of a rigid body in a liquid. The assumptions the author made were unusual in that they allow for a multivalued velocity potential which is possible when the space occupied by the liquid is not simply-connected. He assumed in this way that the space might consist of several mutually unconnected parts, in other words that the body which was moving in the liquid could have cavities filled with liquid. Thus the differential equations he had derived were exceedingly general and could be used for the questions as posed by Dirichlet, Clebsch or Kirchhoff. As regards the method used to derive the equations, it was the most natural and direct one, based on a consideration of the pressures produced by the liquid on the body.

"In the following three chapters, the author was occupied by the integration of the differential equations assuming the absence of force. In the second chapter, having shown what this question reduced to in general, he presented an interesting integration method using series which is applicable when the ratio of the density of liquid to the average density of the body is sufficiently small.

"The third chapter consisted for the most part of the author's own research and was devoted to a consideration of several possible motions. He considered (1) motions with constantly spiralling elements which the author called steady-state; (2) the motions produced when forces are applied to the body whose vector has a definite magnitude and a definite direction with respect to the body and vanishes for a single-valued velocity potential; (3) the

motions of bodies with a symmetry plane that always remains in a fixed plane; and (4) the motions possible for some bodies in which one of the points of the body moves along the central axis of the total momentum vector.

"Cases were considered in the fourth chapter where it turns out that the differential equations can be integrated without any particular assumption about the initial conditions of the motion. Most importantly, the question that was posed and solved here concerned cases when the differential equations permitted a fourth integral, independent of the three well-known integrals that had been demonstrated by Kirchhoff. This integral had the form of an entire homogeneous function of first or second degree variables. The same question had already been studied by Clebsch who had shown the three cases to be unique for which the fourth integral existed. However, due to a mistake in Clebsch's analysis, his opinion that these cases were unique was incorrect. The author was the first to draw attention to this, and in the work in question indicated both Clebsch's mistake and a new case in which a second-order fourth integral exists. The case was also a new one in that the differential equations of interest could be reduced to quadrature. Not content with this integration, and, restricting himself to the solution of the above question, he plunged into a summary of the analyses of Halphen and Kötter who investigated two of the cases discovered by Clebsch. This took the majority of the fourth chapter.

"The motion of a rigid body was considered in the fifth chapter when it is assumed that gravity acts on the body and liquid. This chapter was entirely the author's own work. He reduced the differential equations of motion to a simple form, demonstrating some of the integrals these equations admit for any rigid body. Having made the common assumptions for these bodies, he went on to show some of the interesting motions possible, and the possible directions for several bodies which are in a uniformly accelerated helical motion with a vertical helical axis. He also showed the special family of motions which are possible for bodies having a plane of symmetry that always remains fixed in a vertical plane. Finally, the complete integration of the differential equations of mo-

tion was considered for certain bodies. Given particular assumptions, the integration reduced to solving a system of linear first-order differential equations with three unknown functions which had, by the nature of the equations, to be entire and transcendental. The author did not investigate these functions in general, but limited himself by showing how, having defined the functions, to complete the integration. He then turned to the special case where the total momentum vector is vertical and the functions in question can be expressed linearly in terms of sines and cosines of an argument proportional to the square of the time variable. The investigation of this case concluded the chapter.

"Even from this brief summary it should be clear that the work being considered was distinguished by both a wealth of material and an abundance of new conclusions, many of which were scientifically very significant. Of the last, I must point out the author's discovery of the new case of integrability when forces are absent. This result is so important that it seems to me, and I am not mistaken, it is the most important work that has been done in the field since Clebsch, i.e. for the last twenty years.

"The general integration methods, indicated by the author and applied to cases when the ratio of liquid density and the average density of the body was sufficiently small, is also exceedingly important.

"Finally, I must draw attention to the study in the fifth chapter mentioned above, as it is one of the finest sections of the work and was concerned with the complete integration of the differential equations of motion under gravity...."

In the autumn of 1893, Steklov was invited to become a teacher of theoretical mechanics at the Kharkov Technological Institute. He became an extraordinary professor in the department of mechanics at Kharkov University in 1896, where he lectured on the theory of elasticity [XXXII, op. 1, No. 103, No. 104], theoretical mechanics [130], linear differential equations with variable coefficients [XXXII, op. 1, No. 114] and so on. He had already become known as a mathematician.

The start of the friendship of the Steklovs and Lyapunovs dates from approximately this time and the intimate relationship between their two families continued

unwaveringly until the Lyapunovs died in 1918. Steklov later wrote, "This was the time we drew close to my dear teacher Aleksandr Mikhailovich Lyapunov and his wife Nataliya Rafailovna as well as her father Rafail Mikhailovich and her mother Ekaterina Vasilevna Lyapunov. None of them are alive today*. We were often at each others' places and Olga and Nataliya Rafailovna became quite close." Steklov, during this period, was not only studying mathematics and mechanics but also trying to understand the reality of life directly around him. A number of notes and articles by Steklov on the problems that worried him indicated this. These included an analysis of the price of bread [128, 129], critiques of contemporary Russian literature and some interesting notes on the history of culture. He also wrote articles on the education of women in Russia and on moral problems, as well as making copious notes for critiques of the work of philosophers such as Kant, Hegel, Lluis and Hartmann.

Everywhere he went he would visit museums, libraries and places of historical interest. He was a great connoisseur and lover of music and singing and would do anything to go to the theatre or concerts.

In one of his small notebooks Steklov wrote, "Yes, science is the greatest driving force for mankind and its moral educator! It makes you forget the many evils inherent in man and places him higher than his usual place, makes you see beyond every trifle as would a real man, always reminding him by its own supremacy and fascination of his own high dignity. Yes, poetry and science (and is it not poetic?) uplifts one ...." [XXXII, op. 3, No. 175].

From about 1895 Steklov began to prefer the study of mathematical physics [14, 16, 18, 20-27, 30-46, etc.] and until the end of his life this was the field he was most concerned with and which contained his most important work. However, he returned to mechanics repeatedly [47, 52, 64-66, 70, 71, 77] and significantly expanded the theme of his earlier investigations.

Steklov, even in the Kharkov period of his career had established a correspondence with many scientists both in Russia and abroad, including some eminent ones such

* This note apparently dates to 1921.

as A. M. Lyapunov, N. E. Zhukovskii, N. N. Saltykov, H. Poincaré, C. Jordan, D. Hilbert, J. Hadamard, T. Levi-Civita, Ch. Picard, A. Haar, E. Landau, V. Volterra, S. Zaremba, A. Korn, and A. Kneser [XXXVII]. In the Leningrad Branch of the Archives of the USSR Academy of Sciences (LOAAN) many letters are preserved from scientists, both at home and abroad, addressed to Steklov, as are some of the rough drafts of his own letters. They contain extensive material on many mathematical and mechanics topics. Various other archives and private individuals (such as the son of A. Kneser and one of the authors of this essay) preserve the originals of many of Steklov's letters to his correspondents, at home and abroad. All these letters are interesting, both for their scientific content and the historical material they contain.

Steklov's correspondence with foreign scientists promoted the development, comparison and correction of various methods for solving some problem, or one like it, and led to others being posed and solved. Debates arose from the exchange of letters in which the contribution to science of one or other scholar was objectively appraised. Finally, these scientific links kept him in the mainstream of the latest achievements of science all over the world.

Replying to a letter from L. Raffy, a professor at Paris University, about some work by G. Robin that had been published posthumously by Raffy, Steklov wrote, "The main goal of my studies is to generalise the intricate methods of your illustrious friend, Mr. Robin. All of his studies, I venture to say his classical studies, interest me unequivocally and to the highest degree possible" [XXXII, op. 2, No. 375, p. 7].

In 1901 Steklov wrote to Korn that he had seen almost the same results in one of Korn's works as in one of his own that he had despatched to E. Picard for his journal. He wrote, "Well we obviously can't read each other's thoughts, you and I, but this is not the first time they have coincided, as you well know. I can only welcome our scientific unanimity and the success we have made of solving this bit of mathematical physics, having depended on our dear colleague Mr. S. Zaremba's latest work whose contributions are very important" [XXXII, op. 2, No. 189, pp. 35, 36].

The distinguished Polish mathematician Zaremba whose work was closely related to Steklov's research, wrote, "I am happy as you to see that our works support each other in their approach to the problems posed by physics and which are so fascinating" [XXXII, op. 2, No. 152, pp. 16-19].

Some letters* from N. N. Saltykov to Steklov and Lyapunov contain some information about how mathematics and mechanics was taught at the Universities of Paris, Berlin, Gettingen and other European cities. They are also informative about the relations that existed between foreign scientists and Russian mathematicians and mathematical societies.

In one of his letters to Steklov, dated 6th December 1900, Zaremba wrote, "Reading the interesting notes relating to Laplace's equations you have just published in CR** has made me want to read other works you have written on this question. This is especially important for me since I am lecturing this year on Dirichlet's problem and aspects related to it" [XXXII, op. 2, No. 152, p. 1]. In another letter, dated 10th October 1901, he wrote, "Thank you very much for the interesting information in your last letter and I warmly congratulate you on the wide dissemination you have succeeded in getting for the theory of fundamental functions" [XXXII, op. 2, No. 152, p. 22].

In his letter of 29th November 1901, Steklov wrote to Kneser that, "The Kharkov Mathematical Society has commissioned me to inform you that it has unanimously elected you a member of the Society at its meeting on 29th November 1901, new calendar.***

"Our Society would be very grateful if you would be so kind as to send one of your works, which are always very interesting, for our journal, 'Communications of the Kharkov Mathematical Society'.

* *LOAAN* (the Leningrad Branch of the Archives of the USSR Academy of Sciences), f. 162, op. 2, No. 413; f. 257, op. 1, No. 53.

** CR—*Comptes rendus des séances de l'Académie des sciences de Paris.*

*** A. Kneser was elected a member of the Mathematical Society on 16th November 1901 (old calendar) on the nomination of V. A. Steklov. (*Proc. Khark. Math. Soc.* and *Comm. Khark. Math. Soc.*; 2nd series, 1902, V. 17, p. 292.)

"We publish our Communications, as you may already be aware, in three languages, Russian, French and German..." [XXXVII, p. 25].

Kneser, in a reply dated 5th December 1901, wrote, "May I express my most sincere gratitude to you and all the members of the Society, for my election to the Kharkov Mathematical Society. Your communication has given me great pleasure and I hope, as a member of your Society, to maintain and refresh my links with Russian mathematicians, links that are so valuable for me..." [XXXVII, p. 25-26].

Later on when he was abroad on business or at mathematical conferences some of which were held in Russia, Steklov was able to meet many of these mathematicians in person.

Steklov and his wife used to travel to Moscow at every convenient opportunity to visit his mother and sisters. In 1894 he was greatly saddened by the death of his mother who was then 51. All three sisters had finished gymnasium by that time to become students.

In 1900 the elder sister, Vera, married A. N. Sizemskii, someone the Steklovs knew as a student. A. N. Sizemskii was a member of the "Narodnaya Volya" ("People's Freedom" movement) and had been in prison and exile. From 1900 he was serving in Moscow. After Vera's marriage all three sisters lived together in the Sizemskiis' flat in Moscow.

In 1901 Vladimir Andreevich and Olga Nikolaevna Steklov suffered a terrible bereavement when their ten-year-old only daughter died. Her death deeply affected Steklov and he couldn't apply himself to scientific work for nearly half a year.

In 1902 he obtained his Doctor's degree in Applied Mathematics after defending his thesis *General Methods for Solving Basic Problems in Mathematical Physics* [44]. Shortly afterwards he was elevated to Ordinary Professor of Kharkov University and had nearly forty-five works in print by then. Between 1902 and 1906 Steklov was chairman of the Kharkov Mathematical Society having taken over this post from Lyapunov who had moved to St. Petersburg upon his election to the Academy in 1902.

In 1902 the Petersburg Academy of Sciences elected Steklov a Corresponding Member.

Steklov's extraordinary skill as a teacher and administrator of student affairs was becoming noticeable even in the Kharkov period [XVII]. In 1893 Lyapunov, who was lecturing on Theoretical Mechanics at Kharkov Technological Institute, had to relinquish this course because of his many other activities. On his recommendation, the Institute invited the young Assistant Professor, Steklov, to replace him. In December 1893 the Director of the Institute, Prof. V. L. Kirpichev, wrote about the matter to the trustee of the Kharkov educational district, "A. M. Lyapunov has indicated that Assistant Professor V. A. Steklov is someone who would be able to replace him successfully. I too must testify on his behalf. Mr. Steklov has, by his many contributions and reports at the Kharkov Mathematical Society, proved himself to be a very talented and prominent scientist. He had distinguished himself in various branches of analytical mechanics, particularly hydrodynamics and the theory of elasticity which is of paramount importance in the scientific education of technologists. It really seems very desirable to give this vacancy that has opened up to him" [X]. The quality of his lectures at the Kharkov Technological Institute can be judged from a lithographed lecture course 'Theoretical Mechanics'* that has been preserved [130]. The lecture course is a wonderful introduction to the mechanics necessary for future scientists and engineers. It contains several additional topics on mathematics that were not at that time included in the accepted programme, but which are nevertheless essential for a thorough study of mechanics, viz. elementary vector algebra and vector analysis, an introduction to line integrals, etc. The presentation of mechanics using vector algebra and vetor analysis was a new and very uncommon occurrence at that time.

On Steklov's suggestion the Kharkov Technological Institute introduced practical lessons in place of the so-called 'mocks' (occasional exams). During these lessons, questions which fostered an interest in the subjects being studied were answered and the more difficult parts

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* The lectures are deposited both in the library of the Lenin Polytechnical Institute in Kharkov and in the Saltykov-Shchedrin State Public Library in Leningrad.

of the course were explained further. These practicals have been maintained up to the present day and their usefulness is very well known.

Steklov also took an active part in the political life of Kharkov University. Together with other professors he fought against the University's 1884 charter that was then in force. He worked energetically in the University Council and in the various debates on University reforms.

The atmosphere in the family was permeated with the memory of Nikolai Dobrolyubov. His life was one of genuine audacity and in no small way nurtured a great sense of citizenship in Steklov from his earliest years. Every task he undertook, whether it was large or small, he pursued with great diligence, persistence and care. Both in University life and in the Mathematical Society Steklov did not restrict himself to a simple presence at sessions but worked indefatigably, proposing and compiling drafts, lecture notes, special opinions, etc. He always showed great erudition and independence and had definite opinions.

In 1904 Steklov was elected Dean of the Mathematics Faculty at Kharkov University, taking an even more active part in University life, working on the new University charter and so on.

In the stormy year of 1905 when a revolution flared up in Russia, political speeches also began within Kharkov University. More than a thousand students were present at political meetings held on the 7th and 9th February, and the University became a centre of revolutionary agitation. We can find out about the position Steklov took in relation to these developments from notes of his which are preserved in the Kharkov and Leningrad archives, and also from other sources. Together with the Rector and the other Deans of the University he took active measures to prevent bloodshed amongst the student body.

In a resolution dated 30th January 1906, the Council of Petersburg University decided to petition the Ministry for Public Education to confirm V. A. Steklov as an Ordinary Professor in the Department of Pure Mathematics*. Soon after (5th April 1906) the order was granted by the

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* *Reports of the St. Petersburg University Council for 1906*, St. Petersburg, 1907, No. 62, pp. 15, 47.

Civil Department for Steklov to transfer from Kharkov to St. Petersburg University as an Ordinary Professor. He settled in the 'Petrogradside' district in a small flat at 6/8 Zverinskaya Street where he remained until 1919.

Until the arrival of Steklov, A. N. Korkin (1837-1908) had supervised the mathematical training of students at Petersburg University. However, he was already unable to continue supervision due to his health and age. Although the eminent mathematician A. A. Markov (1856-1922) had lectured at the University he did his basic work at the Academy of Sciences and Lyapunov had only worked at the Academy. Consequently Steklov's presence at Petersburg University was immediately marked by a number of innovations, particularly the introduction of regular practicals. A group of talented students organised themselves quite quickly around Steklov who supervised their studies. Steklov was lecturing in this period on ordinary differential equations [138, 140], partial differential equations [139, 141] and some other subjects (see [137] as well as [118, 119]).

Steklov, who from his student days had preferred to work on his own, required the same independence from his students. He actively supported their scientific societies and publishing committees that issued lectures given by the University's professors.

A review by V. I. Smirnov, a pupil of Steklov, characterises his teaching abilities thus, "It wasn't just those who profited by V. A. Steklov's direct guidance, but also, I think, many of the other students at that time remembered his lectures. He didn't like skating over the generalities of the methods and aims of mathematics, he preferred to show it in action. He did it in such a way that the students were left with an impression not just of separate theorems and terms but of all of them together as a whole. V. A. achieved this by using notes that were short but very valuable when he was proving theorems and solving examples. I think what was especially memorable for the audiences of his lectures on partial differential equations was that he made us aware of several of the contemporary methods and problems of mathematical physics.

"What he required from himself he also demanded from others. He expected work from his own immediate stu-

dents that was within their abilities but was unconditionally independent from the outset. But together with this he would not allow narrow specialisation without an adequately broad mathematical training. Many people today still remember the large demands he made in master's examinations. However, I firmly believe that all those who passed through that ordeal would now be grateful to V. A. since they would realise how much they benefited from the work he made them do. For all the numerous demands he made, it must be said that no exam gave as much satisfaction and simple pleasure as a master's examination under V. A. There wasn't the slightest trace of pettiness in his questions and they were posed in such a way that the examinee did not feel as if he were being tested but rather that he was simply a participant in a conversation. Often some of us argued with V. A. Unfailingly calm he would hear the arguer out and dissuade him if necessary" [VIII, p. 17].

Steklov's qualities as a teacher are commented on, again favourably, by a younger colleague, Ya. V. Uspenskii (1883-1947). "It was to a great extent the original and widely captivating topics of his course that immediately drew so many gifted young people around him. Not only did they work under V. A.'s spontaneous leadership in the University, but continued to do so after they had finished the course... Since he was himself talented he liked other talented people and was solicitously concerned for them, always ready to help. He was never jealous of the success of others and whenever a pupil of his successfully solved a problem better than he could himself, he was the first to endeavour to get the corresponding work published..." [VII, p. 854-855].

In addition to his scientific work and teaching activities at Petersburg University Steklov became occupied with the problems of running the University. His attention to the work that he had begun in Kharkov is reflected in notes [132-136] which are contained in the *Proceedings of the Conference on Drafting a University Charter under the Ministry of Public Education* (1906).

Barely a year had gone by at his new place of work when Steklov proposed at a meeting of the Petersburg University Council that the University should withdraw from the elections of the University's and Academy of

Science's representative to the State Council. He opposed the election on the grounds that this was a "bureaucratic and class institution and only protected the interests of the bureaucracy and privileged classes. The presence of a representative of science in this body would not only be useless but dangerous as well."* From this and other proposals, supplementary notes and reports, it is clear that Steklov was a courageous, righteous and principled man who was not afraid to protest against hypocracy and injustice. He categorically spoke out against the introduction of a professional disciplinary court into Petersburg University and he protested against the admittance of the police into the University building and the arrests they made during student meetings. Those who remained to work towards a professorship were prohibited from lecturing, Steklov suggested rapidly abolishing this prohibition. His proposals were frank and direct and in a number of cases contained patent criticisms of the existing order in Russia. In those years, the years of reaction, such proposals required great courage and fortitude.

In 1910 Steklov was elected adjunct to the Academy of Sciences and an Ordinary Member in 1912. However, he did not forsake his or the University's work. It was only after his election to the leadership of the Academy of Sciences in 1916, that he began to scale down his activities at the University, and after his election as Vice-President of the Academy in 1919 he ceased lecturing. Henceforward he was to work strenuously in the Academy until the end of his life.

Throughout his life, as mentioned above, Steklov continued a voluminous correspondence with many scientists. Starting in 1908 and until he died he corresponded regularly with N. M. Krylov. In the first letter dated 13th September 1908 [XXXII, op. 2, No. 216, p. 3] Krylov wrote that he was studying the expansion of arbitrary functions in the fundamental functions of Poincaré, Zaremba, etc. He requested Steklov to send him Steklov's own works that were printed in the various Russian mathematics journals.

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* *Reports of the St. Petersburg University Council for 1907*, St. Petersburg, 1908, No. 63, pp. 13-16.

In the next letter [XXXII, op. 2, No. 216, pp. 7-8]*, Krylov wrote that he would be unable to obtain an exceedingly important work Steklov had published in a French journal. A detailed presentation of the work was to appear in an Italian journal that was poorly distributed in Russia, and begged Steklov to send him a reprint.

Further on he wrote that his scientific interests had recently been in the very field mentioned above and that a study of Steklov's significant work relating to the theory of fundamental functions had been very useful to him. He also wrote that he had sent Steklov two papers noting that they contained results that were far from final that could be generalised. He asked Steklov to send him his remarks which, he believed, would be a positive help in his own work.

In a letter dated 29th March 1910 [XXXII, op. 2, No. 216, p. 4] Krylov thanked Steklov for informing him about the dispatch of Steklov's work about expanding arbitrary functions in series. He also expressed satisfaction that his method, which was based on an appropriate application of Ritz's theorem and suitable for all hypergeometric polynomials, had, when applied to Jacobi polynomials, given results that were the same as Steklov's; this was a complete guarantee of the truth of their correctness.

Further on Krylov wrote that he would be very interested to know whether J. D. Tamarkin was applying his results to some types of non-selfconjugate fourth-order equations. He was interested because he had been told personally by Dini during a conversation with him in November 1909 about his wide ranging memoir which would appear before long apropos these equations.

In another letter of 18th October 1911 Krylov wrote, "Dear Professor Vladimir Andreevich, Permit me to send you the enclosed copy of my inaugural lecture (2nd May 1911) which was printed in the 'Proceedings of Kiev University for 1911'. I would be very happy to get a reprint from you of the work you have done on series expansion. It is a subject of great interest to me and I had the pleasure of talking with you about this work during my last visit to you this spring." [XXXII, op. 2, No. 216, p. 10.]

* Written in French.

As can be seen from this letter Steklov and Krylov not only were carrying on a lively correspondence, but had also met personally and discussed various scientific problems.*

"Dear Vladimir Andreevich," Krylov wrote in a letter dated 17th October 1914 [XXXII, op. 2, No. 216, pp. 11-12]. "I am sending you a note of Ritz's method in the same letter with a request. If you find it possible, would you print it in the 'Proceedings of the Academy' (small size). Forgive me for bothering you but in view of the external situation and the disruption of international correspondence, I can't turn to those people who I've approached before..."

On the 20th November 1915 he wrote, "Dear Vladimir Andreevich, Further to the conversation we had the evening I had the pleasure of presenting to you my work 'On Convergence of Quadratures', I am now able to add the following...." He produced several explanations and further on wrote, "...and then by introducing your subsidiary functions it was obvious how the result could be generalised. An application to various sorts of quadrature seems interesting especially to Markov's quadrature. Yakov Viktorovich [Uspenskii], who has told me about the existence of a memoir by Markov and to whom I sent my own results relating to their convergence, has on his part informed me of a more general result some time ago, namely that convergence does not occur for all continuous functions...."

Between 1918 and 1925 Steklov suffered blows other than the death of his wife. In 1918 A. M. and N. P. Lyapunov passed away, in 1922 Academician A. A. Markov died, and in 1925 Steklov's most outstanding pupil, A. A. Fridman, died unexpectedly at the age of 37.

Steklov, on the 16th December 1915 [XXXII, op. 2, No. 216, p. 59], wrote, "Dear Nikolai Mitrofanovich! I was unable to write earlier because I was occupied with various non-scientific affairs. Now I can briefly share

* Unfortunately we were unable to find any of Steklov's letters written to Krylov during this period. The Leningrad Branch of the Archives of the USSR Acad. Sc. [XXXII, op. 2, No. 216] has a few of his letters, apparently rough drafts, and belonging to a later period (starting about 1915) and the Moscow Archives of the Academy has only one letter, f. 689, op. 4, No. 117, pp. 1-4.

with you some results which will be contained in a memoir that is to appear next year. I demonstrate that a direct application of the methods presented in my memoir* yields the following results:

"1. The convergence of every quadrature formula is established if its coefficients satisfy the condition

$$\sum_{k=1}^{n} |A_k^{(n)}| \leqslant K$$

"2. The convergence of every formula of mechanical quadrature is established when the coefficients $A_k^{(n)}$ are positive for any integrable** function in a given interval (a generalisation of Stieltjes's theorem for Gauss's formula).

"3. The exact expression of the remainder term R_n is given for every quadrature formula (whatever the coefficients A_k might be), if of course the function $f(x)$ permits derivatives of various orders ...."

In another letter [XXXII, op. 2, No. 216, pp. 60-62] (undated***) Steklov wrote, "Dear Nikolai Mitrofanovich, I am sending you the proof of the theorem that interested you...." He went on to discuss a formula for a mechanical quadrature with positive coefficients assuming that it converges for any continuous function. He deduced several properties of the ordinate $X_k^{(n)}$ distribution in the quadrature formula from this condition. He also demonstrated that every quadrature formula with positive coefficients would converge for every integrable function if it converges for a continuous function.

On the 7th January 1916 [XXXII, op. 2, No. 216, pp. 22-23] Krylov wrote, "Dear Vladimir Andreevich, Following our yesterday's telephone conversation when you told me Weierstrass's theorem could be proved using elementary properties of Legendre polynomials and their completeness equation, I was thinking about this last night. I came to the conclusion that a very simple proof can be given based on the assumption that every function of bounded variation can be expanded into a series in Legendre polynomials.... The proof does not presuppose an

* [98].

** Riemann integrable.

*** But apparently written in 1915.

equation of completeness and an objection to the use of the theorem on the Legendre polynomial expansion can be parried by saying that the Legendre polynomials are a particular case and trigonometric polynomials could have been taken instead. Then the expansion theorem can be proved for every function that can be represented in the form of integrals, something utterly elementary and follows from your memoir...."*

2

AN ORGANISER OF SOVIET SCIENCE

Real changes in the Academy of Sciences followed on the heels of the February Revolution. In May 1917 the Imperial Academy of Sciences was renamed as the All-Russian Academy of Sciences and for the first time in its history it elected its President. Academician A. P. Karpinsky became the first President of the Academy, but in fact had been governing it since May 1916 whilst acting as Vice-President. Academician Karpinsky, a geologist and scientist of world renown, had shown himself to be an excellent organiser of science.

From about the middle of 1917 a group of scientists formed within the Academy who were trying to bring science closer to the people and to strengthen its influence on society. This trend, the inspirer of which was A. M. Gorky, had Steklov as an active participant. A man with strong convictions, courageous and audacious, a genuine scientist and superb organiser, a believer in the greatness of the future of the Russian people and science, Steklov was one of those scientists who put the scientific world of the country on the road to the establishment of a new social order. The election of such a man as this to the Vice-Presidency of the Academy (1919) was an important event in its history. The massive burden of anxieties, resulting from the growth and revivication of the Academy added to those of all science throughout Russia, laid on Steklov's shoulders.

Immediately after the Great October Socialist Revolution, the Bolshevik Party headed by Lenin required the wide use of science for the erection of the new social order

* He had memoir [95] in mind.

of socialism. In April 1919 Lenin pointed out that "We must take the entire culture that capitalism left behind and build socialism with it. We must take all its science, technology and art. Without these we shall be unable to build communist society. But these science, technology and art are in the hands and in the heads of the experts"*. Lenin entrusted the Narkompros** with the task of attracting the scientists working at the Academy to the construction of culture and the economy. A. V. Lunacharskii, the People's Commissar for Education in the RSFSR, wrote "The Narkompros has a direct order from V. I. Lenin with relation to the Academy, viz. carefully and cautiously, only gradually and without injuring its organs, to bring it into the new communist framework more solidly and organically."***

Steklov did not vacillate a minute and took the side of the conquering people and carried many scientists with him. The Bolsheviks and Soviet Power gradually guided the strength of the scientists working at the Academy to the building of the new society. Not only did this strengthen and expand the Academy, but it also led to a genuine flourishing of all Soviet science. The activities of eminent scientists such as A. P. Karpinskii, V. A. Steklov, I. P. Pavlov, K. A. Timiryazev, S. F. Oldenburg and A. M. Krylov were important in this context.

The young Soviet republic had then to live through some difficult years. However, even in the darkest hours of the Civil War, scientific work was not stopped in spite of the great difficulties.

Striving to expand research on physical mathematics, Steklov together with Academicians Markov and Krylov proposed in January 1919 that a specialised Mathematical Cabinet be organised within the Academy. Its functions were formulated in a note which the scientists brought to the attention of the Physico-Mathematical Branch.****

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* V. I. Lenin, *Collected Works*, Vol. 29, p. 70.

** Narkompros stands for the People's Commissariat for Education.—*Tr.*

*** A. V. Lunacharskii, "Academy of Sciences and Soviet Power", in: *Rabochaya Gazeta*, 14th August 1925.

**** *Proc. Phys.-Math. Branch of Russian Acad. Sc.*, 1919, Sec. 22.

The Mathematical Cabinet (it was awarded the names of P. L. Chebyshev and A. M. Lyapunov) began its work in 1919 and was chaired by Steklov who donated his personal library to it.

In January 1921 Steklov presented the Physico-Mathematical Branch a detailed note in which he substantiated the pressing necessity for a physico-mathematical institute to be established under the Academy of Sciences*. Following in M. V. Lomonosov's wake, he wrote, "...The light that can be shed on a science by a scientist initiated into the mysteries of mathematics can be seen clearly from those few major natural sciences that use it. Everything in these sciences that was dark, doubtful and incorrect, mathematics has made transparent, brilliant and true. Not one of the natural sciences, whose affairs are concerned with real creativity and not the mere collection of raw material, can manage without mathematics, the mother of science. Just consider physics, the first amongst the sciences, as it is in the plan for the Lomonosov Institute. At the present time mathematics and physics are so fused together it is often difficult to distinguish where mathematics begins and the physics ends." Steklov concluded, "If this idea was not sufficiently appreciated two hundred years ago, today it is essential to recognise it, even in a wider form, as axiomatic."

Steklov's proposal found unanimous support at the Academy and in 1921 the Physico-Mathematical Institute of the Russian Academy of Sciences was founded in view of "the imminent requirement in science intimately to connect the physical sciences with pure mathematics". Steklov was elected its Director and the Mathematics Cabinet, the Physics Laboratory and the Seismic Network were incorporated into the Institute and appropriate departments were organised around them. The Physico-Mathematical Institute existed until 1934 when on the 11th March it was divided into two independent institutes: the V. A. Steklov** Mathematical Institute headed by Academician I. M. Vinogradov, and the P. K. Lebe-

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* *Proc. Phys.-Math. Branch of Acad. Sc.*, 1921. Appendix (10th January 1921).

** The Physico-Mathematical Institute of the USSR Academy of Sciences was given Steklov's name immediately after his death in 1926.

dev Physics Institute under the direction of Academician S. I. Vavilov.*

The scale of scientific research in the country expanded considerably once the Civil War was ended, as can be seen from the activities of the Academy of Sciences. On 27th January 1921 Lenin received Gorky, Academicians Steklov and Oldenburg, the head of the Academy of Military Medicine, V. N. Tonkov, when they had a discussion "on the provision of scientific research work in the Soviet republic".

Recalling this interview with Lenin, Gorky later wrote, "I remember I was with three members of the Academy of Sciences.** The conversation was about the necessity to reorganise one of the higher scientific institutions in Petersburg. Talking about the scientists he said with satisfaction, 'Them I understand. They are clever men. Everything is simple and strictly formulated and you can see immediately they know very well what they want. To work in this way is a pleasure. I like very much this...' He named one of the biggest names in Russian science and the next day he told me over the phone,' Ask S.*** if he is going to work with us. ' When S. accepted the suggestion, Lenin was genuinely pleased and rubbing his hands together, he joked, 'Look, one after the other, we will pull in every Russian and European Archimedes and then the world whether it likes it or not will be turned upside down' " [XXXV, p. 383].

This commendation by Lenin graphically characterises Steklov's activities after the October Revolution. Once introduced, Steklov met Lenin on other occasions and Steklov spoke highly of Lenin. He wrote, "Vladimir Ilich seemed to me to be the paragon of a political activist combining the ability to act decisively and unflinchingly to achieve his aims with the rare gift of political intuition, which enabled him to divine by an odour, so to speak, those elemental beginnings which move the life of a people and for which the usual measures for a logical

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* Protocol of the Presidium of the USSR Academy of Sciences, 11th March 1934.

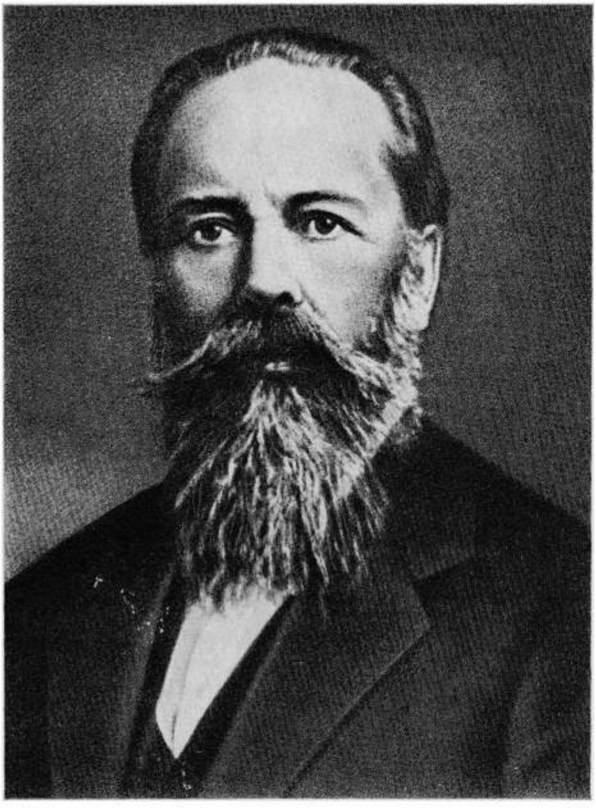
** V. N. Tonkov, a well known anatomist, was not at that time a full member of the Academy. He became a full member of the Academy of Medical Sciences in 1944.

*** Steklov.

Steklov the gymnasist

Steklov's wife and daughter (1890's)

Steklov a Professor at Kharkov University



Steklov the President of the USSR Academy of Sciences

A reproduction from a picture by V. A. Serov. From left to right: Vice-President of the Academy of Sciences, Academician V. A. Steklov, the head of the Academy of Military Medicine, Professor V. N. Tonkov, the permanent secretary of the Academy of Sciences, S. F. Oldenburg, and standing behind V. I. Lenin, the writer A. M. Gorky. The meeting took place on the 27th January 1921.

and reasonable construction are almost never suitable. As a practical statesman he has, in my view, the rare ability to bring a desired aim into reality, choosing the most appropriate means for a given time and place.

"This precision of thought and activity puts him above the other political leaders and naturally singles him out as the leader of the revolution which could only have taken place under his leadership" [XXXII, op. 3, No. 184, p. 1], [LX].

Steklov also met the People's Commissar for Education, Lunacharskii, to discuss the affairs of science. A note to Steklov has been preserved in which M. F. Andreeva wrote, "Dear Vladimir Andreevich, since Anatolii Vasilevich Lunacharskii can only get to Aleksei Maksimovich's by 9 o'clock this evening, Friday 19th April 1918, Aleksei Maksimovich Gorky asks if you would please come to us at 9 o'clock in order to meet and converse with Lunacharskii. Aleksei Maksimovich and I send our greetings and await you this evening. Maria Andreeva" [XXXII, op. 2].

After Steklov's death, Lunacharskii remembered, "In his attitude to science, it was as if Steklov had been destined to become our ally... His interview with Lenin had a colossal effect on the deceased. Steklov was always enthusiastic about it, but even Lenin after that conversation with Steklov and Oldenburg said to me, 'Serious people, things are always possible and necessary with them'" [XLIX].

Steklov was one of the initiators and organisers of the Special Provisional Committee on Science under the USSR Council of People's Commissars. It consisted of representatives from the Academy and the People's Commissariats and with Steklov's immediate participation produced and implemented governmental decrees. These promoted the growth of the Academy itself and also the country's many other scientific establishments.

During the difficult years of the Civil War, Steklov was actively studying the Kursk magnetic anomaly which led to the discovery of a massive deposit of iron one. He supervised the theoretical and calculational part of the expedition to the northern region of the Kursk magnetic anomaly, [XXXII, op. 3, No. 15, p. 18], to the study of which Lenin had given great significance. In appreciation of Steklov's work on this anomaly, the USSR Coun-

cil for Work and Defense advertised their thanks to him in April 1923 [XXXII, op. 3, No. 1, p. 48].

Steklov was a member of the Standing Commission for the study of tropical countries and also active in two committees, the Committee for the Affairs of the Central All-Russian Astronomical Observatory and the Committee for the All-Russian Hydrology Institute. He took an active part in the restoration and construction of the network of seismic stations in the country and intimately concerned himself with the preparation of the new charter for the Academy of Sciences. The Charter was adopted almost without amendments in 1927, barely a year after Steklov's passing. He had a great influence on the building of the Academy, being a member of the Construction Commission. Finally he was a member of the Academy of Sciences Publishing Commission and took part in the International Commission for the publication of the works of Leonard Euler, the permanent seat of which was in Zurich, Switzerland.

Only a most general and a far from complete picture of Steklov's activities after the October Socialist Revolution can be put together from this short list of his responsibilities. At the same time he was engaged in a great deal of social and administrative work within the Academy of Sciences. It should be noted however that he was not content to be a member of a commission by name only and would show himself to be an energetic figure of great initiative in whatever he participated.

Recalling Steklov's activity as Vice-President of the Academy of Sciences in those difficult years, Krylov wrote, "Thanks to Steklov and the trust he had earned from the Soviet government, the Zoological and Ethnographic Museums' collections were preserved from destruction (from damp and meagre heating). It is impossible to estimate their worth in monetary terms. All the tribulations relating to a fitting celebration of the 200th Jubilee of the Academy of Sciences was borne by Steklov. The Academy's new 1927 charter were framed by him personally and ratified by the government virtually without change" [XIX]. Throughout his life Steklov was energetic and active in every possible social matter both to do with science and enlightenment. He was unafraid of difficulties and did not lose heart before them.

Remembering his teacher, V. I. Smirnov wrote of Steklov, "V. A. took on both administrative work and the organisation of science at a moment when it seemed nothing could be done and everything was crumbling. But it was not in his temperament to fold his arms in a difficult moment. The tighter the corner and the greater the energy V. A. would fight for something..." [VIII, p. 18]. Elsewhere he wrote, "Vladimir Andreevich was a man with a tremendous will and purpose and social temperament. This is clear from his activities... But science was first in his life... It would be erroneous only to consider him a mathematician. He was a great connoisseur of Russian history and music. His habit on various occasions of citing cases from Russian history and quoting Peter the Great, Lomonosov and Lobachevskii was not simply a love of Russian style but an expression of a blood-deep relationship with all Russian culture..." [XXIV].

Steklov's work as Vice-President of the USSR Academy of Sciences was clearly the reflection of the profound patriotism of this authoritative Russian scientist and his concern for the development of Soviet science.

3

THE LAST YEARS OR HIS LIFE

After his election to the Vice-Presidency of the Academy of Sciences, Steklov moved to a flat in the 'Academician's Building' on the banks of the Neva close to the Lieutenant Shmidt Bridge. Chebyshev, Lyapunov, Markov, Karpinskii, Oldenburg and other Academicians had also lived in the building. The large flat (No. 1) Steklov occupied was on the first floor and looked out onto the Neva.

Steklov's home was always open to close friends and a multitude of scientists. Gorky was a frequent visitor.

Olga Nikolaevna always surrounded her husband with her care and if perhaps during the difficult years she sometimes suffered from malnutrition, she hid this from her husband. However, it all told on her health and in 1920 Steklov saw her off to Kislovodsk where she died on 7th September aged 59. His wife's death was a tragedy

for Steklov, he became more introverted and stern, and his personality began to show a sombreness unusual for him.

Although keeping up his administrative tasks, Steklov continued in his final days to be intensively occupied with science, preferring as before to work into the night. He brought out several books, two on mathematical physics [118], [119], one on Lomonosov [147] and one on Galileo [148], then the titles *Mathematics and Its Significance for Mankind* [149] and *To America and Back, Impressions* [151], he also wrote a textbook on differential equations [127] and various academic articles [125], [126], etc.

Despite his many occupations, Steklov found time for scientific visits and maintained his wide relations with renowned scientists abroad.

The main event during Steklov's last years was his trip to the International Mathematics Congress held in Toronto, Canada, in August 1924. In his book [151] Steklov wrote, "I had been sent by the Academy of Sciences and the Narkompros* to America, England, France, Germany and Spain to participate in international conferences on mathematics and geophysics, and also to become generally acquainted with the state of science abroad. The chief aim was to attend the International Mathematics Congress in Toronto, Canada, which was taking place at the same time as the General Meeting of the British Association."

Steklov presented two papers. *On the Problems of Representing Functions Using Polynomials, Approximation of Definite Integrals, Expansion of Functions into Series in Polynomials and Interpolation from the Standpoint of Chebyshev's Ideas* [121] and *About a Posthumous Work by A. M. Lyapunov on the Equilibrium Forms of a Rotating Heterogenous Liquid* [122]. The other scientists accompanying Steklov to the Congress from the Soviet Union were N. M. Gyunter, V. A. Kostitsyn, N. M. Krylov, and Ya. V. Uspenskii who made a number of reports on the theory of numbers, interpolation, integral equations and hydrodynamics among others.

Steklov's paper about Lyapunov's posthumously printed work made a great impression on many of the parti-

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* The People's Commissariat for Education.—*Tr.*

cipants at the congress. He later wrote, "Not only were the methods and results of Lyapunov's posthumous work new to the West European and American scientists, but so were many of the conclusions he had reached since 1916. It is indisputable that after my paper, and especially once our Academy publishes Lyapunov's posthumous work, interest in his classical investigations will grow stupendously. Many foreign scientists will have to revise their own work, and several of them will find it to be very incomplete or out of date, even though they might have appeared after the publication of Lyapunov's above mentioned work" [151, pp. 27-28].

As to the theme of the Congress Steklov wrote, "Although the Congress carried the designation 'Mathematical', it is clear from the list and topics of the sessions that the designation was to be understood in its widest sense" [151]. In fact, not only were there sessions on pure mathematics but also on philosophy, history, and didactics as these related to mathematics and the application of mathematical methods to engineering and even economics. Such a wide approach to the discussion of mathematical problems was rated very highly by Steklov who remarked that it was percolating even wider through the natural sciences. He wrote, "...The Professor of Zoology at St. Andrews University, Scotland, D'Arcy Thompson presented a paper in the session on geometry entitled 'Repeating Forms of Regular Polygons and Their Relations to Archimedean Bodies'. It was an aspect of mathematics to which he had been led in his researches on zoology. I use this as an example of how even sciences like zoology are passing on from a platonic respect for mathematics to its application and are discovering in the meantime their desire to become a real science, even though it be in the distant future" [151, p. 27].

We can now say with pride that Steklov's prophetic words have come to pass, mathematics has become simply an instrument of academic research, not only say in chemistry or biology, but also in many other sciences that are quite remote from it such as medicine, economics, ecology, philology, and music.

At the same congress Steklov met its president, Prof. J. Fields. Digressing slightly it was at this conference that Prof. Fields proposed two gold medals with prizes be

awarded to young mathematicians who had made outstanding contributions to mathematics. The proposal was implemented in 1932 after his death. During the middle of the week devoted to the Congress, Steklov was one of the eight scientists to be elevated as an honorary doctor of Toronto University, the ceremony taking place on the 13th August.

When the congress finished Steklov stayed in Canada a further three weeks, spending the majority of the time travelling around the country. Together with some of the other participants at the congress he went from Toronto to the Island of Vancouver, returning back to Toronto by another route. On the 7th September 1924 Steklov left Toronto for Chicago staying there for several days. Then he went on to New York and on the 12th September boarded the liner 'Olympic' for London.

Several days later Steklov was in London. On the 27th September he left for Paris in order to attend a session of the Paris Academy of Sciences with Academician Krylov. At the Academy they were courteously met by the Permanent Secretary, the well-known mathematician E. Picard and the renowned scientist P. Appel who was the President of the Academy. Both were extremely interested to learn about the work of the Toronto Congress since neither had been able to go themselves. Steklov's paper on Lyapunov's posthumous work was particularly interesting to them.

Four days later Steklov travelled from Paris to Berlin and thence, via Riga, home. While in Berlin he had to clear up some misunderstandings regarding deliveries of instruments and equipment to the USSR Academy of Sciences. Later Steklov was to write, "At the start I had wanted to visit Vienna, Halle, Leipzig, Goettingen and Hamburg as well as Berlin and to participate in the meeting of seismologists in Innsbruck as a member of the German Seismological Society. However, an unusual flood required that I hasten to Leningrad and I was unable to carry out this itinerary.... After a forty-eight hour stop in Riga I left for Moscow and returned to a Leningrad on 17th October 1924 still suffering from the flood of the 23rd September" [151].

In June and October 1925 Steklov visited scientific institutions in Germany, Italy, and Austria. He met leading

scientists there and held talks about how to strengthen scientific contacts. In Germany in particular he had long conversations with the Permanent Secretary of the Berlin Academy of Sciences, M. Plank. These travels of Steklov abroad and his participation at the Congress in Toronto with other Soviet scientists, enhanced the authority and influence of Soviet mathematics abroad.

In one of his letters [XXXII, op. 2, No. 216, p. 58] (undated, though it seems to have been written between 1915 and 1917) Steklov wrote, "Dear Nikolai Mitrofanovich, I am conveying to you, as promised, the following significant result. Every quadrature formula converges for any function $f(x)$ that satisfies the following condition:

$$\left| \frac{1}{h} \int\limits_{x}^{x+h} (f(z) - f(x))\, dz \right| < \varepsilon(h)$$

for any x in the interval $(-1, 1)$ (for simplicity), where $\varepsilon(h)$ is independent of x and tends to zero together with h ...."

In 1925, two hundred years had passed since the foundation of the Academy of Sciences. Long before this date the Academy had begun to prepare for the celebration and Steklov devoted a lot of time and energy to make it not just a jubilee, but a celebration of Soviet science in general. On the eve of the jubilee he repeatedly got articles into the pages of the press describing the Academy's long life story. Thus in one such article, '200 Years of the Academy of Sciences' [183], Steklov refuted the unsubstantiated assertions of the overseas press that the Academy of Sciences had lost its scientific credibility and independence under Soviet rule. He cited a host of examples that showed the growth and spread of the Academy's institutions as a result of the aid and assistance of the Soviet government. During the jubilee he participated in the ceremonies and receptions thrown in honour of foreign guests, repeatedly making speeches and giving addresses. A note [179] by Steklov gives a short historical survey of the activities of the Academy over two hundred years and characterises the work of individual scientists, L. Euler, M. V. Lomonosov, V. V. Petrov, H. F. I. Lenz, C. G. J. Jacobi, P. L. Chebyshev, B. B. Go-

litsyn, and some others. Article [180] contains his speech at the formal party given by the Leningrad Gubispolkom* during the jubilee.

Steklov's correspondence with foreign scientists remained lively. In particular, he was able at this time to renew his correspondence, disrupted by the First World War, with the authoritative German mathematician A. Kneser, mentioned before. On the 9th January 1925 Kneser wrote to Steklov, "Dearest Colleague, It is with gratitude that I thank you for your communication of 5th December last year. It would be a great honour for me to become a Correspondent** of your long famous Academy and renew my close contacts with Russian mathematicians ..." [XXXVII, p. 56].

The jubilee session of the Academy took place between the 5th and 14th September 1925 in Moscow and Leningrad. On one of those days Steklov with a group of Academicians went round the Kremlin. All that day it rained very heavily, and Steklov, who had only a light raincoat and no umbrella, got wet through. The following day he felt very unwell but, as always, carried on regardless. The consequences proved fatal.

Very soon afterwards Steklov went to Italy, returning home in December 1925. However, his health had not improved.

The 23rd February 1926 marked the hundredth anniversary of N. I. Lobachevskii's discovery of non-Euclidean geometry. Steklov had a high regard for the scientist and attended the ceremony in Kazan. The journey to Kazan produced a considerable deterioration in his condition, and travelling from Moscow to Kazan in a freezing carriage, he caught a chill. He returned to Moscow with a temperature of around 39°C but as soon as his temperature disappeared he returned to work. He did not want a doctor and only after many pleas did he agree in April to consult one. He only lasted another month,

* Provincial Executive Committee.—*Tr.*

** A. Kneser was elected a Corresponding Member of the All-Russian Academy of Sciences on the 6th December 1924, having been nominated by V. Steklov, P. Lazarev, and A. Belopolskii. Cf. Extract from the Proceedings of the Academy of Sciences. Appendices. *Proc. Russian Acad. Sc.*, Series 6, 1924, V. 18, pp. 428, 452-453.

his health deteriorating the whole time and a shortwindedness appearing.

Steklov's last public appearance was on the 21st April 1926 at a formal session of the USSR Academy of Sciences in honour of the participants in Amundsen's expedition leaving for the North Pole. In his speech, according to Academician Oldenburg, Steklov "expressed an opinion that had guided his whole life concerning the proximity of science to the working masses. He felt that every scientific discovery or achievement was significant only so long as people had a use for it ..." [XLV].

Steklov decided in May 1926 to leave for the Crimea for treatment. Before he left he called upon the Academy on the 14th May, it was to be his last visit. He left for the Crimea on the 15th May intending to stay for a month but fate was to decree otherwise. On the 30th May in Gaspra, Vladimir Andreevich Steklov breathed his last. He was buried in the Volkov Cemetery in Leningrad.

Two days after Steklov's death, the People's Commissar Lunacharskii wrote, "The Soviet government, proletariat and peasantry of the USSR must honour the person of V. A. Steklov, a representative of science, who set himself the task, straight after the Revolution, of reconciling the intelligentsia to the new order and of setting common work with the Soviet government" [XLVIII].

On the same day (1st June 1926) Academician A. E. Fersman wrote, "His idea was that science should be exalted, and he made this the basis of his whole life. He was the first one bold enough to put before our Academy the idea of a truly international union of scientists at a time when the ripples of the conflicts between individual nations had overwhelmed the scientific community of the West" [LIX]. That day P. P. Lazarev wrote, "...the memory of Steklov both as the scientist who has created important and wide fields of modern mathematics and as an administrator and figure in society, will live for a long time not only at the Academy of Sciences but throughout the Union of Soviet Socialist Republics for whose culture Steklov has done so much" [XLVI].

In Steklov's obituary, carried in the journal *Nature* [LVIII] we read, "V. A. Steklov was one of the few in that brilliant pleiads of Russian mathematicians who, like Lobachevskii, Ostrogradskii, Chebyshev, and Lya-

punov, have opened new pathways in defiance of the established and ossified forms. Like the great Euler, he constituted an entire stage in the development of mathematical ideas, his practical mind could grasp the tenuous connection between an abstract idea and a concrete phenomenon in physics or geophysics. In a like manner, his mathematical studies were done both to develop mathematics and to solve problems in mechanics and molecular physics. His enthusiasm for geophysics put him at the forefront of seismic affairs in the country, and his interest in the movement of air brought him into contact with the urgent problems of aviation, whilst the laws of gravity led him to an analysis of the most difficult questions of geology about the ice ages". V. I. Smirnov, in his note 'In Memory of Acad. V. A. Steklov' [LVII], wrote, "A man of uncommon energy and initiative, he seemed to be involved in everything he was engaged in. He reanimated the educational life of St. Petersburg University by his appearance and collected young people around him who were starting to study science and became a centre of the scientific and mathematical life".

The well known German mathematician Kneser expressed himself thus about Steklov, "Steklov's personality made a strong and pleasing impression on everybody who came into contact with him. It was a happy combination of a highly educated man of European culture and the original spirit peculiar to those of his race. He had widely and deeply studied not only science, but also music and was interested in other forms of art. During the journeys to Europe and America he undertook over the last few years as a spokesman of his country's science, he represented it in the best possible way. Those around him esteemed him most for his seething energy and practical mind in everything related to the organisation of scientific work. He won the trust of the new government of his country, repeatedly travelling to Moscow in order to mediate between the scientific world and the representatives of the new government. Steklov rendered an extraordinary service to Russian science by this and really reestablished scientific work after the Revolution and Civil War, chiefly in the institutes connected with the Academy of Sciences" [IX, p. 109], cf. also [XXXVII, pp. 60, 61].

4
HIS WORK ON MATHEMATICAL PHYSICS

Mathematical physics is a topic concerned with the construction and investigation of mathematical models of physical phenomena.

Since Newton's time mathematical physics was developing in parallel with physics and mathematics. Classical mathematical physics considered the problems of classical physics and mechanics such as the oscillations of elastic bodies and media, thermal conductivity and diffusion, electrodynamics, optics, fluid dynamics, transport phenomena, the theory of potentials, and the theory of stability. Over the last half-century, the scope and methods of mathematical physics have expanded considerably. This has been due to the emergence of quantum physics, the theory of relativity and new problems in gas dynamics, transport theory and plasma physics, as well as the enlistment of a wide arsenal of mathematical techniques including computers.

Classical mathematical physics was concerned above all with the differential equations which describe physical processes. In order to describe a process fully over time it is necessary, firstly, to give a picture of the process at some initial moment of time (initial conditions) and, secondly, to specify the regime at the boundaries of the domain in which the process occurs (boundary conditions). The system of differential equations together with their initial and boundary conditions is known as the boundary-value problem in mathematical physics. The boundary-value problem is indeed a mathematical model of the physical process, and classical mathematical physics was fully occupied by its study.

The following second-order linear differential equations are typical of mathematical physics.

The *vibration equation*

$$\rho \frac{\partial^2 u}{\partial t^2} = \operatorname{div}(p \operatorname{grad} u) - qu + F \qquad (1)$$

The *heat (diffusion) equation*

$$\rho \frac{\partial u}{\partial t} = \operatorname{div}(p \operatorname{grad} u) - qu + F \qquad (2)$$

The *steady-state equation*

$$-\operatorname{div}(p \operatorname{grad} u) + qu = F \tag{3}$$

For each equation $\rho > 0$, $p > 0$, $q > 0$, where ρ, p, q, and F are known functions, their physical sense being defined by the process being studied.*

In particular, when ρ and p are constants and $q = 0$ equations (1), (2) and (3) simplify and become:

$$\frac{\partial^2 u}{\partial t^2} = a^2 \Delta u + f \tag{4}$$

which is the *wave equation*;

$$\frac{\partial u}{\partial t} = a^2 \Delta u + f \tag{5}$$

which is the *heat conduction equation*; and

$$\Delta u = -f \tag{6}$$

which is *Poisson's equation*. In each case $a^2 = p/\rho$, $f = F/\rho$ and

$$\Delta = \operatorname{div} \operatorname{grad} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

which is the Laplace operator.

Initial conditions. At some initial moment in time, say $t = 0$, both the value of the disturbance, u, and its rate of change over time, u_t, are given for the vibration equation (1). Thus we have:

$$\begin{aligned} u(x, y, z, 0) &= \phi_0(x, y, z) \\ u_t(x, y, z, 0) &= \phi_1(x, y, z) \end{aligned} \tag{7}$$

At time $t = 0$, only the temperature u is given for the diffusion equation (2), i.e.

$$u(x, y, z, 0) = \phi_0(x, y, z) \tag{8}$$

Naturally the steady-state equation (3) needs no initial conditions.

Boundary conditions. Let D be a domain in which the process takes place and let S be its boundary. For practi-

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* The derivation of these basic equations and the formulation of boundary-value problems of mathematical physics can be found in almost every textbook on the subject. See, for example, V. S. Vladimirov, *Basic Equations of Mathematical Physics*, Mir Publishers, Moscow, 1984.

cal purposes it is sufficient to consider only piecewise smooth surfaces S. Finally, let $n = n(x, y, z)$ be the external normal to S at the point (x, y, z). The following types of boundary conditions describe the regime at the boundary for practically all real physical processes:

$$u|_S = g \tag{9}$$

here the value of u itself is given at the boundary S for all $t \geqslant 0$;

$$\left.\frac{\partial u}{\partial n} + hu\right|_S = g, \quad h \geqslant 0 \tag{10}$$

here a linear combination of u and $\frac{du}{dn}$ is given at S for all $t \geqslant 0$; in particular, when $h = 0$ the flux of u is given.

The boundary-value problem for *Laplace's equation*, $\Delta u = 0$, with the boundary conditions as given in (9), is known as *Dirichlet's problem*. The boundary-value problem with the following boundary condition

$$\left.\frac{\partial u}{\partial n}\right|_S = g \tag{11}$$

is known as *Neumann's problem**. If the domain D is bounded, then a distinction can be made between the interior and exterior Dirichlet and Neumann problems. The exterior boundary-value problem requires an additional condition to be satisfied at infinity. When the number of variables is $n \geqslant 3$, the solution must tend to zero, but if $n = 2$ the solution must be bounded at infinity.

Note that there can be any number of spatial variables in the above equations and boundary conditions, in particular the number is less than three. This occurs for linear domains (strings and rods) or two-dimensional domains (membranes), as well as when all the quantities which describe the process under consideration are independent of some spatial variables.

Steklov clearly recognised that the boundary-value problems of mathematical physics were the models of physical phenomena. He acknowledged as such in these extracts from his book *Basic Problems of Mathematical Physics* [118].

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* Steklov called this the basic problem of hydrodynamics.

"Solving problems of mathematical physics reduces to determining one or more quantities, which characterise a physical process taking place in a given medium (or body), as functions of the position of every point of the medium and time using one or more differential equations.

"These equations are derived from a small number of the simplest (possibly) hypotheses. These hypotheses lie at the basis of the theory of each physical phenomenon and are presented as the results of a generalisation of a long chain of experiments and observations on physical processes that actually take place in nature or are created artificially.

"As a result of these abstractions, or generalisations, a small number of basic statements (or hypotheses) can be formulated which must be mutually independent and not in contradiction to any of the facts of reality known at the time.

"The hypotheses underlie the theory of a physical phenomenon and the whole theory is developed deductively therefrom using the axioms of mathematics and the basic laws of general mechanics and according to the methods of differential and integral calculus.

"Thus the differential equations which characterise the nature of a considered process for every point of the medium and every moment of time are constructed according to the physical data of problem.

"A problem reduces to the determination, as a function of time, of both the coordinates of every point in the medium in which the phenomenon being considered occurs and the quantities which define the physical properties of the medium, of the unknowns present in the differential equations obtained. That is, the problem reduces to the integration of those equations.

"The solution so obtained must agree with all the data which were collected by direct observations of the process" (p. 43).

"The essence of physical processes is not known to us in every detail. When we are generalising all the data from experiments and observations, we are building, as was mentioned above, several of the most likely hypotheses and use them to create in our imagination a special mechanical model of the process. The more fully the processes that are described by this model correspond to the facts that can be directly observed from the phenomena

which actually occur in nature and which in turn are being substituted for by our model, the better the model, and the more confidence we can have in the hypotheses that were originally used in constructing the model.

"Having created a model that is the simplest possible and whose mechanical construction is known to us, we can then express its laws of motion in an analytical form using the principles of mathematics and general mechanics. The mathematical relations obtained this way do not strictly speaking describe the motions that occur in nature, rather they describe those which occur and must occur in the model we constructed. Having derived analytically various properties and features of the motions of our model, we compare them with reality. If they constantly coincide with reality and if new facts derived from the known properties of our constructed model are confirmed by experiment and observation, then the link between our artificial construction and the actual physical phenomenon becomes more trustworthy, and the hypotheses that were put at the basis of our reasoning become more and more probable, turning, over the course of time, into laws.

"If, on the other hand, even one of the conclusions from the analytical formulae, which express the laws of motions in the frame of our constructed model, turns out to be in clear contradiction with the data of direct observation, then the model must be considered inadequate and the hypotheses (or some of them) on which it was constructed, considered unsatisfactory and irreconciliable with reality. A new model must then be constructed or the old one altered appropriately.

"The entire history of experimental science, particularly the exact branches like geometry, mechanics, physics and astronomy, has been a pattern of the creation and constant reconstruction of such models.

"As we apply the above said to the problems of mathematical physics we are interested in, we must above all else note the following. If the differential equations, together with the initial and boundary conditions mentioned above, are formed on error-free foundations and are not in clear contradiction to reality, then they must give a unique and definite answer to each problem, just as in general mechanics differential equations with certain

initial conditions must give a unique and definite solution.

"In the real world we observe that every physical process is a result of definite causes and always has a definite and unique course. A material body, for example, that is located in a definite position in space and set in motion with a definite velocity under the action of given forces, may have one and only one definite motion.

"*Thus the primary and necessary condition to make the motions in our constructed models and those in the real physical processes we want to describe using the mechanical models correspond, is that the above differential equations, together with the initial and boundary conditions, yield a unique and definite solution*" (pp. 54, 55).

Steklov published more than 150 works, the majority of which were related to various aspects of mathematical physics. His first work on the subject appeared in the 1890's. At that time new ideas were beginning to penetrate into mathematical physics due to the efforts of H. A. Schwarz, E. Picard, and especially, H. Poincaré. Classical mathematical physics which had been created at the end of the 18th century and in the first half of the 19th by L. Euler, J. L. D'Alembert, J. Bernoulli, J. L. Lagrange, J. Fourier, P. S. Laplace, S. D. Poisson, J. Liouville, and M. V. Ostrogradskii was not satisfactorily meeting the newer, higher demands on rigour. Thus it became necessary both to justify the old system theoretically and to create new methods which would lead to rigorous solutions of the problems. Primarily this concerned a number of problems related to the rigorous justification of the so-called *method of separating the variables.* Bernoulli, D'Alembert, Euler and Lagrange had all attempted to solve the problem on the vibration of a homogeneous elastic string by the method of separating the variables even in the 18th century.* At the beginning of the 19th century the method of separating variables was worked out in detail and applied to the conduction of heat in solids by the French mathematician Fourier in his celebrated work *La Theòrie Analytique de la Chaleur* (1822) (*The Analytical Theory of Heat*). Subsequently the

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* For this reason Steklov also called Fourier's method the Euler-Bernoulli method.

method of separating variables became known as Fourier's method. Note that M. V. Ostrogradskii (1802-1862) succeeded in significantly developing the rudiments of Fourier's investigation of heat conduction.

We will now present Fourier's method as applied to the problem of the free vibrations of a string with ends at 0 and l.

Equation (1), initial condition (7), and boundary condition (10) take the following forms:

$$\rho u_{tt} = (pu_x)_x - qu, \quad 0 < x < l, \quad t > 0 \tag{12}$$

$$u(x, 0) = \phi_0(x), \quad u_t(x, 0) = \phi_1(x), \quad 0 \leqslant x \leqslant l \tag{13}$$

$$u_x - hu|_{x=0} = u_x + Hu|_{x=l} = 0, \quad t \geqslant 0 \tag{14}$$

We will assume that the functions ρ, p and q in equation (12) and the constants h and H in boundary condition (14) are independent of time t (this assumption is essential for Fourier's method). The constants h and H are considered to be non-negative.

We are looking for a solution, u, of equation (12) in the form of a product of the function X, which is solely dependent on the spatial variable x, and a function T, which is solely dependent on time t, i.e.

$$u(x, t) = X(x) \cdot T(t) \tag{15}$$

Substituting (15) into equation (12) we obtain:

$$\rho X T'' = (pX')' \, T - qXT$$

which gives

$$\frac{T''(t)}{T(t)} = \frac{[p(x) X'(x)]' - q(x) X(x)}{\rho(x) X(x)} \tag{16}$$

The left-hand side of this equation is independent of x and the right-hand side of t. But this means that neither side is dependent on either x or t, i.e. they are constants. Let us call this constant λ. Equation (16) can now be transformed into two equalities as follows:

$$T''(t) + \lambda T(t) = 0 \tag{17}$$

$$-(pX')' + qX = \lambda \rho X \tag{18}$$

Thus, the partial differential equation (12) has, as a result of substitution of (15), been separated into two ordinary differential equations, (17) and (18). It is said in

this case that the *variables have been separated.* As a result of this separation an additional (unknown) parameter λ has arisen.

Returning to equation (18) we are interested in a solution that will satisfy boundary condition (14), viz.

$$X'(0) - hX(0) = X'(l) + HX(l) = 0 \tag{19}$$

Clearly, boundary-value problem (18)-(19) will always have the trivial solution, i.e. $X = 0$. Such solution is uninteresting and we will exclude it from our discussions from now on. It may turn out that equation (18) has a non-trivial solution, which satisfies boundary conditions (19) for some λ. These λ are known as the *eigenvalues* of boundary-value problem (18)-(19) and the corresponding solutions, as the *eigenfunctions* of the problem. The boundary-value problem (18)-(19) about finding eigenvalues and eigenfunctions is known as the *Sturm-Liouville problem.*

It can be shown that there exists a countable set of non-negative eigenvalues, i.e.

$$0 \leqslant \lambda_1 < \lambda_2 < \ldots < \lambda_k, \ldots, \lambda_k \to \infty \text{ as } k \to \infty$$

There is a unique eigenfunction $X_k(x)$ corresponding to each eigenvalue λ_k, and the eigenfunctions are mutually orthogonal with weight ρ so that

$$\int_0^l \rho(x) X_k(x) X_i(x)\, dx = \delta_{ki} \tag{20}$$

where δ_{ki} is the Kronecker delta: $\delta_{ki} = \begin{cases} 1, & k = i \\ 0, & k \neq i \end{cases}$.

As an example we will derive the eigenvalues and eigenfunctions of the Sturm-Liouville problem, i.e. (18)-(19), setting $\rho = 1$, $p = 1$, $q = 0$, $h = \infty$ and $H = \infty$. Thus we get:

$$-X'' = \lambda X, \quad X(0) = X(l) = 0 \tag{21}$$

$$\lambda_k = \frac{k^2\pi^2}{l^2}, \quad X_k(x) = \sqrt{\frac{2}{l}} \sin \frac{k\pi}{l}, \quad k = 1, 2, \ldots \tag{22}$$

Graphs of the eigenfunctions $X_k(x)$ for $k = 1, 2, 3$ are given in Fig. 1 as bold curves.

If $\lambda = \lambda_k$ the general solution of equation (17) has the form

$$T_k(t) = a_k \cos \sqrt{\lambda_k}\, t + b_k \sin \sqrt{\lambda_k}\, t \tag{23}$$

where a_k and b_k are arbitrary constants.

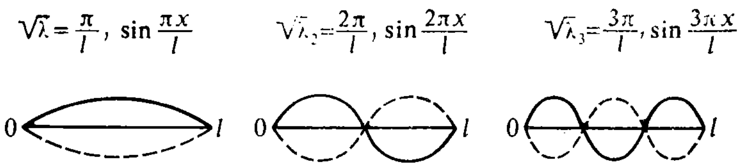

Fig. 1

Thus owing to (15) we have constructed a countable number of linearly independent solutions to equation (12) of the form:

$$u_k(x,\ t) = (a_k \cos \sqrt{\lambda_k}\, t$$
$$+ b_k \sin \sqrt{\lambda_k}\, t)\, X_k(x), \quad k = 1,\ 2,\ \ldots \tag{24}$$

Each of these solutions satisfies boundary conditions (14) Hoping to obtain a solution to the problem described by (12)-(14), we compile a formal series

$$u(x,\ t) \sim \sum_{k=1}^{\infty} u_k(x,\ t)$$
$$= \sum_{k=1}^{\infty} (a_k \cos \sqrt{\lambda_k}\, t + b_k \sin \sqrt{\lambda_k}\, t)\, X_k(x) \tag{25}$$

and choose the unknowns a_k and b_k so that they formally satisfy initial condition (13). Thus we must have:

$$\phi_0(x) = \sum_{k=1}^{\infty} a_k X_k(x), \quad \phi_1(x) = \sum_{k=1}^{\infty} \sqrt{\lambda_k}\, b_k X_k(x) \tag{26}$$

It follows from the properties of normalisation and orthogonality (20) of eigenfunctions $X_k(x)$ that the coefficients a_k and b_k in (26) are uniquely defined by:

$$a_k = \int_0^l \rho \phi_0 X_k\, dx, \quad b_k = \frac{1}{\sqrt{\lambda_k}} \int_0^l \rho \phi_1 X_k\, dx \tag{27}$$

Thus the formal series (25), which we call the *formal solution* of problem (12)-(14), is well defined.

Each term, $u_k(x, t)$, of series (25) is a harmonic vibration and represents a *standing wave with a frequency* $\sqrt{\lambda_k}$. The sequence of numbers $\sqrt{\lambda_1}, \sqrt{\lambda_2}, \sqrt{\lambda_3}, \ldots$ is called the *spectrum of the natural frequencies* of the vibrating string. The harmonic vibration with the least frequency is called the *fundamental tone* and the other vibrations are called the *overtones*. Solution (25) is the sum of the harmonic vibrations corresponding to separate tones (fundamental and subsequent overtones) and their summed action creates the *timbre* of the note produced by the string. The standing waves for a homogeneous string with fixed ends are shown in Fig. 1 for $k = 1, 2, 3$. Analogously, for the boundary-value problem with

$$\rho u_t = (pu_x)_x - qu, \quad 0 < x < l, \quad t > 0 \tag{28}$$

$$u(x, 0) = \phi_0(x), \quad 0 \leqslant x \leqslant l, \tag{29}$$

$$u_x - hu|_{x=0} = u_x + Hu|_{x=l} = 0, \quad t > 0 \tag{14}$$

(h and H being non-negative constants) we obtain a corresponding formal series (solution) which has the form

$$u(x, t) \sim \sum_{k=1}^{\infty} a_k e^{-\lambda_k t} X_k(x) \tag{30}$$

where the a_k are defined by the first formula in (27).

The formal scheme presented of Fourier's method for solving boundary-value problems can be seen to be in need of justification at the following points. It must be proved that:

(1) there is an infinite (countable) number of eigenvalues and eigenfunctions (problem A);

(2) the eigenfunctions are 'sufficiently numerous', i.e. that every 'sufficiently good' function can be expanded into a convergent Fourier series in eigenfunctions (problem B); and

(3) the formal series obtained, (25), converges and yields a solution (possibly generalised) to the original problem (problem C).

The same problems arise of course when justifying Fourier's method for multidimensional cases too.

These basic problems had clearly come to a head at the end of the 19th century and their solution was then very

difficult. They attracted the attention of such a leading mathematician as Poincaré. In two of his memoirs, *Sur les équations de la physique mathématique* (1894) and *La méthode de Neumann et le problème de Dirichlet* (1896), Poincaré formulated the basic results of his investigations in this direction. However, his analysis rested on the assumption that the harmonic function that solved Dirichlet's problem had regular normal derivatives on the boundary of a given surface and that these derivatives existed for a double layer potential too and that they coincided on the surface (cf. [18]).

A proof that the Dirichlet, Neumann and Robin problems had solutions which avoided the difficulties encountered by Poincaré and moreover with a minimal number of assumptions about the surface S and the boundary function g is precisely what Steklov dealt with between 1897 and 1902. He systematically expounded his results in his thesis *General Methods of Solving Basic Problems of Mathematical Physics* [44], and in a monograph *Basic Problems of Mathematical Physics* Pt. II, [119].

In the first work [18] (1897) (also cf. [27]) on Neumann's problem, Steklov proposed another approach. He presented the solution of the interior Neumann problem in the form of a simple layer potential on the surface S:

$$u(x, y, z) = \frac{1}{2\pi} \iint\limits_S \frac{\mu(x', y', z')}{r} \, ds \tag{31}$$

μ is an unknown density and r is the distance between the points (x, y, z) and (x', y', z'). He found the density μ by using successive approximations, i.e.

$$\mu = \rho_0 + \rho_1 + \rho_2 + \ldots + \rho_k + \ldots \tag{32}$$

where

$$\rho_k = -\frac{1}{2\pi} \iint\limits_S \rho_{k-1} \frac{\cos\psi}{r^2} \, ds, \quad \rho_0 = g, \quad k = 1, 2, \ldots \tag{33}$$

and ψ is the angle between the external normal to S at the point (x, y, z) and the vector from (x, y, z) to (x', y', z') (Fig. 2) (Steklov actually chose $\psi' = \pi - \psi$).

It follows from the recurrence formula (33) and Gauss's formula that

$$\iint_S \rho_k \, ds = \iint_S \rho_0 \, ds, \quad k = 1, 2, \ldots \tag{34}$$

It should be noted that by about this time Lyapunov had rigorously proved, for the first time, the properties

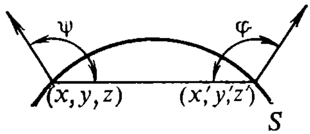

Fig. 2

of simple and double layer potentials for the class of surfaces Steklov called *Lyapunov surfaces**. There is a proof of Lyapunov's basic theorems in Chapter 1 of [119]. In particular, the sufficient condition for the density is obtained under which the double layer potential has regular normal derivatives from the inside and from the outside of the surface (Lyapunov's fourth theorem). An example was given showing that this condition cannot be improved.

We will say, following Steklov, that the Lyapunov surface S satisfies *Robin's principle* if the function ρ_k defined in (33) satisfies, for every point of surface S, the following inequality:

$$|\rho_k - \rho_{k-1}| < N\tau^k, \quad k = 1, 2, \ldots \tag{35}$$

given some $N = N(\rho_0) > 0$ and $\tau < 1$, and also τ is independent of the choice of the initial function ρ_0. It now follows from Robin's principle that if the function $\rho_0 = g$ obeys the condition

$$\iint_S \rho_0 \, ds = \iint_S g \, ds = 0, \tag{36}$$

then

* For a definition of Lyapunov surfaces see, for example, V. S. Vladimirov *Basic Equations of Mathematical Physics*, Mir Publishers, Moscow, 1984.

$$|\rho_k| < N\tau^k, \quad k=1,\ 2,\ \ldots \tag{37}$$

Note that condition (36) must be necessarily fulfilled for the solution of the interior Neumann problem, and thus estimate (37) remains true. It follows from this that series (32) converges absolutely and uniformly on S defining the continuous density μ. The existence of a solution to the interior Neumann problem and its representability in the form of a simple layer potential were proved by this.

According to the theory of integral equations, formulae (32) and (33) are none other than successive approximation for Fredholm's integral equation:

$$\mu = -\frac{1}{2\pi}\iint\limits_S \mu \frac{\cos\psi}{r^2}\,ds + g \tag{38}$$

In this $+1$ is the characteristic number of the conjugate kernel $-\frac{\cos\mu}{2\pi r^2}$ (the angle φ is shown in Fig. 2) and $\mu \equiv 1$ which is the corresponding eigenfunction (due to Gauss's formula). In this way the solubility condition (36) of integral equation (38) comes out in accordance with Fredholm's theorems. The solution of this homogeneous integral equation is determined to within the eigenfunction μ_0:

$$\mu_0 = -\frac{1}{2\pi}\iint\limits_S \mu_0 \frac{\cos\psi}{r^2}\,ds \tag{39}$$

It can be shown that the eigenfunction μ_0 is unique; it is known as the *Robin potential density*, and the corresponding potential of a simple layer is called the *Robin potential.* It follows directly from (39) that the Robin potential is a constant inside the domain D and is non-zero. From this, and also due to (31), it immediately follows that the solution to the interior Neumann problem is determined to within the arbitrary additive constant.

To find the Robin potential is a basic problem of *electrostatics* or *Robin's problem.*

Steklov gave a solution of this problem in [18, 27], i.e. he constructed the solution of the homogeneous integral equation (39). He constructed this solution using the iteration of (33), the initial function ρ_0 being arbitrary

and positive. The series

$$\lim_{k\to\infty} \rho_k = \rho_1 + (\rho_2 - \rho_1) + (\rho_3 - \rho_2) + \dots$$

then converges absolutely and uniformly on the surface S due to Robin's principle (35) and defines the continuous (positive) * density μ_0 of the Robin potential.

Thus Steklov eliminated the gap in the original variant of Robin's method by proving rigorously that the eigenfunction exists.

We want to emphasise particularly that the convergence of the successive approximation method for integral equation (38) was shown by Steklov for the characteristic number and for all surfaces that satisfy Robins' principle. In the general theory of Fredholm's integral equations the convergence of the successive approximation method is guaranteed if the magnitude of the lowest (in magnitude) characteristic number is greater than unity. Here we have an excellent example of the way the specific character of a problem has been employed!

The exterior Neumann problem and Dirichlet's problem, both interior and exterior, were solved in an analogous way [27, 34, 44, 53, 119].

The solution of the interior Dirichlet problem was found in the form of a double layer potential,

$$u(x, y, z) = \frac{1}{2\pi} \iint_S \frac{\mu(x', y', z')}{r^2} \cos\phi \, ds$$

with an unknown density μ. The unknown function μ can be given as

$$\mu = \frac{1}{2} [\rho_0' - (\rho_1' - \rho_0') + (\rho_2' - \rho_1') - \dots]$$

where $\rho_0' = g$ and

$$\rho_k' = \frac{1}{2\pi} \iint_S \rho_{k-1}' \frac{\cos\phi}{r^2} ds, \quad k = 1, 2, 3 \dots$$

It was proved that for a surface S which satisfies Robin's principle (35), Neumann's principle is also satisfied:

$$|\rho_k' - \rho_{k-1}'| < N\tau^k, \quad k = 1, 2, 3 \dots \qquad (*)$$

* The function μ_0 is positive for convex surfaces.

and thus the interior Dirichlet problem is soluble for these surfaces in the form of a double layer potential.

Steklov considered, in [119], the problem of the representation of the solution of Neumann's problem as a double layer potential and also the *Gauss problem* about the representation of the solution of Dirichlet's problem as a simple layer potential. The necessary and sufficient condition for the boundary function g was obtained which ensured the existence of a regular normal derivative for the solution of Dirichlet's problem. Finally the simplest problems of mathematical physics which reduce to either Dirichlet's or Neumann's problem were considered, namely problems of the vortex motion of liquids, stationary temperature and the construction of Green functions.

Steklov showed that Robin's principle was obeyed for sufficiently smooth surfaces that do not deviate much from a sphere, and also for convex (sufficiently smooth) surfaces [18, 27, 31, 34, 44]. Finally in 1902 he proved in [53] the validity of Robin's principle for all Lyapunov surfaces using a lemma of S. Zaremba. Thus by 1902 potential theory had been completely justified, largely thanks to Lyapunov's and Steklov's studies.

The substantiation of potential theory entailed a rigorous justification of the Schwarz-Poincaré method* with a proof that the eigenvalues and eigenfunctions of the following boundary-value problem exist

$$-\Delta X=\lambda X, \quad \frac{\partial X}{\partial n}+hX|_S=0, \quad 0\leqslant h=\text{const}\leqslant\infty \tag{40}$$

The Schwarz-Poincaré method is as follows. We are looking for a solution to the boundary-value problem.

$$-\Delta u=\lambda u+f, \quad \frac{\partial u}{\partial n}+hu|_S=0 \tag{41}$$

in the form of a series in powers of λ, viz.

$$u=u_0+\lambda u_1+\lambda^2 u_2+\dots \tag{42}$$

For each u_k in (42) we get a boundary-value problem:

$$-\Delta u_0=f, \quad -\Delta u_k=u_{k-1}, \quad \frac{\partial u_k}{\partial n}+hu_k\Big|_S=0 \tag{43}$$

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* Cf. the memoir of Poincaré's dated 1894 and cited above.

After Steklov's work, the functions u_k could be constructed by the methods of potential theory. Then Poincaré proved that the function u, as defined in series (42), is meromorphic in λ with an infinite number of poles, all its poles $\lambda = \lambda_k$ ($k = 1, 2, 3, \ldots$) being simple and the eigenvalues of the boundary-value problem (40). The residue at the pole λ_k is a linear combination of the eigenfunctions corresponding to the eigenvalue λ_k.

Thus were problems A, B, and C, as formulated above, solved in order to justify Fourier's method (cf. in the same vein [16, 34, 46, 53, 63], his later notes [79, 80], memoir [91], and book [118]).

It should be noted that Steklov introduced the fundamental function V_k as far back as 1895:

$$\Delta V_k = 0 \text{ in } D, \quad \frac{\partial V_k}{\partial n} = \lambda_k \phi V_k \text{ on } S \, (\phi > 0) \qquad (44)$$

which are called the *Steklov fundamental functions* (as distinct from eigenfunctions, the parameter enters here the boundary condition). The existence of the functions was established by Steklov in his doctoral thesis [44] in 1901 and he used them to solve Dirichlet's and Neumann's problems (also cf. [53]). Steklov's fundamental functions are mutually orthogonal on S with weight ϕ and are the extension of spherical functions to the general surfaces.

In modern terms, (44) is a non-trivial example of a problem about the eigenvalues for a pseudo-differential operator defined on surface S.

Lyapunov wrote in his review of Steklov's doctoral thesis [44], "The investigation of problems of mathematical physics that are in some way related to Laplace's equation is the topic of this work. The most important of these are reduced to a definition of functions which satisfy Laplace's equation for the given conditions of continuity and single-valuedness for some boundary condition, which is what every condition that must be fulfilled at the boundary of the considered domain is called. The problems of defining these functions, called harmonic functions, may be utterly different because they depend essentially on the character of the boundary conditions. Of all the problems of this type, two deserve special attention. In one, the boundary conditions reduce to assigning values to the

harmonic function* at the boundary and in the second they reduce to assigning values to its normal derivative**... He devoted special consideration in this work to these two problems. He treated them assuming that the boundary of the domain was a single closed surface which obeys the most general conditions, and that the domain reduced either to the space inside the surface or the space outside it.

"These problems are among those which have been long formulated. They were formulated as far back as the first half of the nineteenth century and are connected with the names Gauss, Green, and Dirichlet. However, such are the difficulties engendered by these problems that for a long time not even their possibility could be established rigorously for any general assumptions, to say nothing of any practically important solutions for them.

"It must however be noted that the basic principle, on which the method is based, which I will call Neumann's principle***, was shown by this scientist only for surfaces which are convex at all points. Thus Dirichlet's principle**** as well had only been shown for this assumption.

"In view of the above, both Neumann himself and other scientists, especially Schwarz, have continued investigating this problem, trying to establish Dirichlet's principle for non-convex surfaces too. But the methods that were proposed by them to attain this aim were extremely complicated and did not possess the desired generality, and up until 1889 when Poincaré published his investigation of the topic, a general proof of Dirichlet's principle did not exist.

"Finally, Poincaré gave such a proof, for a fairly general method he invented allowed Dirichlet's principle to be established utterly independently of whether or not the surface was convex. It should be noted that the method he proposed was based on rather simple considerations.

"Poincaré's investigation was therefore very important. His method did not, however, yield an analytical expression for the unknown function, and in this respect was significantly inferior to Neumann's method.

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* Dirichlet's problem.—*Authors.*

** Neumann's problem.—*Authors.*

*** Cf. (*) on p. 56.—*Authors.*

**** The solubility of Dirichlet's problem.—*Authors.*

"A consequence of this was that a few years later Poincaré pursued a new investigation to try to extend Neumann's method to non-convex surfaces. In 1896 he published a very important memoir *La méthode de Neumann et le problème de Dirichlet* in which he showed that such an extension was really possible....

"As regards the extension of Neumann's method, Poincaré's memoir still could not be considered to resolve finally the question being studied. For all the importance of his ideas he left many points unproved. Thus Poincaré based himself on one transformation of the variables that he thought was always possible, but still it could not be taken without proof.... Thus there was still a very wide field for investigation after the memoir mentioned, an investigation that could take a wide variety of directions....

"As far back as 1886 Robin suggested the method of successive approximations analogous to Neumann's method to solve the problem. However, Robin, like Neumann, proved the convergence of his method for convex surfaces alone and he assumed that the possibility of the problem was beyond any doubt.

"This possibility had always been taken for granted by everyone without proof and Poincaré had incidentally based his conclusions on it in the work mentioned. A long time ago I drew attention to the fact that this point is the weakest part of the theories under consideration and that the existence of the boundary value on the surface for the derivatives of the harmonic functions in Dirichlet's problem, as well as the possibility of the electrostatic problem, requires special investigation.

"In the autumn of 1897 this investigation was taken up by Steklov and myself in two different directions. Steklov quickly succeeded in finding a modification of Robin's analysis for which the convergence of the method could be proved irrespective of the preliminary assumption on the possibility of the problem. Thus there was a proof of the possibility for convex surfaces too....

"Above all it was necessary to revise Poincaré's analysis in order to liberate it from arbitrary assumptions. Steklov noted that everything depends on the proof of one general proposition that lies at the bottom of all Poincaré's conclusions. This proposition, which Steklov called the 'fundamental theorem', was proved by Poincaré with

the help of the above mentioned transformation, and Poincaré in his proof used considerations that assumed the existence of the boundary values of derivatives (they have been mentioned more than once).

"Now as regards Poincaré's transformation, when the author wrote the work under consideration he had not succeeded in eliminating it and because of a lack of any general proof that the transformation was possible, he had to consider the transformation to be a special condition that characterised the surface. On this assumption he proved the 'fundamental theorem'. As to all the other assumptions Poincaré had made, Steklov managed to free the proof from them by giving the theorem in a rather less general form than that in Poincaré's proof. He could do this because the theorem, in this form too, was quite sufficient for the later conclusions.

"The proof of the 'fundamental theorem', together with some subsidiary propositions, made up the first chapter. The basis for the later conclusions was thus established, and the rest, which was contained in the following four chapters could be considered to be proved for all the surfaces that permit Poincaré's transformation.

"It must however be noted that this transformation was necessary only for the proof of the 'fundamental theorem' and did not itself have any further importance. Thus Steklov's conclusions could be extended to every case in which the theorem could in some way be established. This comment is very important since, as will be shown below, the author now has a proof of the 'fundamental theorem' that is completely independent of Poincaré's transformation.

"We now turn to the following chapters. In the second chapter the author above all proved Neumann's principle for the case where the initial function is a simple layer potential with zero mass*. In essence, this is the form of Neumann's principle I depended on in my above mentioned investigation to establish Robin's principle. It only differs in that I had not assumed the mass of the layer to be zero. However, in the case being considered this restriction follows naturally from the proof which was based on the 'fundamental theorem'.

* More accurately, charge (cf. (36)).—*Authors.*

"Then the author continued on to a proof of Robin's principle* using a method I had mentioned in the investigation just named. But the consequence of the above restriction was that he obtained the principle in a form where both the initial and all subsequent densities correspond to a zero mass**. However, the general Robin principle can be easily deduced from this particular form and this was shown by the author. Thus he had established the possibility of both the basic electrostatic problem and Robin's method.

"I should note that having established the general Robin principle it was not at all difficult to generalise Neumann's principle which could immediately be shown for the case where the initial function is such that the double layer potential depending on it has regular normal derivatives on the surface. True, the author did show this generalisation, but he only did so in the fourth chapter after a long series of quite complicated arguments which were aimed at establishing Neumann's method for Dirichlet's problem. Meanwhile had he previously established Neumann's principle in the form just indicated these arguments would have been superfluous since Neumann's method would follow directly from it.

"In the third chapter the author primarily concentrates on the determination of a harmonic function for a given value of its normal derivative on the surface. Both methods for solving the problem were presented, namely one indicated by Neumann and based on his principle, and the other on Robin's principle. Steklov has already drawn attention to this last method in one of his articles where it was presented under a certain assumption concerning the surface. Now he proved it in the general form.

"The author then turned to a problem concerning a steady-state temperature, which can be reduced to the foregoing using series. This important topic in the analytical theory of heat had already been considered by Poincaré in *Sur les équations de la Physique Mathèmatique* in 1894. But due to the absence at that time of proofs for several basic propositions, Poincaré's analysis did not satisfy all

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* Cf. (35).—*Authors.*
** Cf. (34).—*Authors.*

the requirements of rigour and needed even more assumptions. In view of this, the author of the work now being reviewed dwelt anew on this problem which now admits a rigorous solution. The author showed how this could be achieved using both the Schwarz-Poincaré method and a method based on Robin's principle.

"The fourth chapter was in the main devoted to Dirichlet's problem. The author began with a consideration of the known transformation using mutual radius vectors and then combining the transformation with Poincaré's transformation he showed several corollaries, which are related to Green's functions and the general Dirichlet problem, that follow therefrom once the possibility of the basic electrostatic problem had been established. Nevertheless, the propositions did not play a role later and so were given by the author for the sake of completeness.

"The author then went on to prove the validity of Neumann's method for Dirichlet's problem limiting himself to the case where the boundary values of the harmonic function given on the surface satisfied a condition such that the double layer potential dependent on them permitted a regular normal derivative on the surface. This is the condition which, according to my criterion, ensures the existence of normal derivatives on the surface for the unknown harmonic function too....

"Turning to the fifth and final chapter, this is the most important part of the work. It was devoted to the theory of special functions which are a well-known generalisation of spherical and other similar functions and which are called, following a suggestion by Poincaré, fundamental.

"Poincaré had first drawn attention to the possibility of such a generalisation in *La mèthode de Neumann et le problemè de Dirichlet.* He had shown here how important this type of function was for investigating various aspects of finding harmonic functions under given boundary conditions. But Poincaré had not proved that these functions existed and had only sketched out in general terms how their theory should be constructed.

"Two years later the topic of fundamental functions was thoroughly developed by one of Poincaré's students, Le Roy. Starting from another definition of this type of function and using the Schwarz-Poincaré method he proved

the existence of and developed a full theory for his fundamental functions.

"Soon afterwards the author of the work now being considered also undertook an investigation in the same direction and had, as far back as 1895, met similar functions by chance. Having generalised these he arrived at a definition of fundamental functions that was different both from Poincaré's definition and from Le Roy's. Moreover, the same Schwarz-Poincaré method allowed him to prove the existence of the new functions. The result of these investigations was the theory he presented in this work's fifth chapter.

"Le Roy's fundamental functions, like Steklov's, were distinct from Poincaré's functions. Thus the theory of the latter could not be reduced to that of the former. However, after Le Roy's and Steklov's investigations there was no doubt that the theory of Poincaré's functions could be constructed on the same basis. Several indications of this were given by the author of this work.

"Having established his fundamental functions existed and shown a few of their properties, Steklov went on to prove special formulae that gave expressions, in the form of series, for the integrals of the product of two functions extended to a given surface. This kind of formula could have a variety of important applications thanks to its considerable generality. I have shown this for the case of spherical functions when the surface considered is a sphere or ellipsoid.

"In this work the indicated formulae were proved for a general case of fundamental functions using a special method which the author had suggested earlier for another analogous case.

"Having based himself on the above formulae, he showed a large number of varied applications of fundamental functions. Thus he gave expansions into a special series for the potential and attraction of a mass distributed over the surface, and showed how these series could be availed of for determining the potential and attraction at any point in space where the values of the potential were known at the surface. The author also went on to an expansion of any function given at the surface into a series in terms of fundamental functions and showed that such an expansion in which the coefficients were determined accord-

ing to a known rule, was always possible so long as the series obtained turned out to converge uniformly on the surface. Finally, the author showed how to solve Dirichlet's problem using the fundamental functions, and did the same for another fundamental problem in which the harmonic function was defined by the value of the normal derivative on the surface.

"This in outline is the content of the work. I have limited myself to the essential points and not dwelt on the various detailed questions broached on by the author. However, it is clear from even this short exposition how important and varied the contents of this book are. One theory alone, that of fundamental functions, during whose creation the author showed great originality, is in itself a very important contribution to science. No less important, I feel, is his rigorous establishment of the relation between Neumann's principle and the proposition the author called the 'fundamental theorem'.

"I have already indicated above that the author proved this proposition using Poincaré's transformation, but that the remaining conclusions are independent of whether this transformation is possible and are true whenever the 'fundamental theorem' can be proved in some way or other. It can thus be seen how important it was to establish this theorem independently of Poincaré's transformation.

"A few months ago Professor Zaremba of Crackow University published a small article in which he showed that as a result of his previous studies it follows that Neumann's method could be justified without recourse to the above transformation. Soon afterwards, Steklov and Assistant-Professor Korn of Munich University based themselves on this indication by Zaremba and independently arrived at a proof of an inequality that follows from the 'fundamental theorem' and lies at the heart of the proof of Neumann's principle. Recently Steklov has found a proof for the 'fundamental theorem' itself. Thus he has at last managed to establish it without recourse to Poincaré's transformation.

"However this important result could not be included in this work because it was already being printed when he obtained it...."

It should be noted that in [22] (1897) (cf. also [46]) Steklov had found an exact value for the constant in

Poincaré's inequality*: If a function f is once continuously differentiable in a closed domain $\bar{D}$ and satisfies the condition $\iiint\limits_{D} f\,dv = 0$, then

$$\iiint\limits_{D} f^2\,dv \leqslant \frac{1}{C(D)} \iiint\limits_{D} \left[\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2\right] dv \tag{45}$$

where $C(D) = \lambda_1$ is the least (positive) eigenvalue of Neumann's problem for the Laplace operator in domain D. In terms of one variable when $D = (0, l)$, relation (45) becomes (cf. [16, 118]);

$$\int\limits_0^l f^2(x)\,dx \leqslant \left(\frac{l}{\pi}\right)^2 \int\limits_0^l f'^2(x)\,dx \tag{46}$$

Even earlier in 1896 (cf. [14]), Steklov had established the validity of inequality (45) for a function f which satisfies the boundary condition $f\,|_S = 0$ and when the constant $C(D)$ is exact and such that $C(D) = \lambda_1$, which is the least eigenvalue of Dirichlet's problem for the Laplace operator in domain D**.

Many different generalisations of inequality (45) exist and inequalities of this type are widely used in modern mathematics, they are the simplest examples of the so-called *embedding theorems* of functional spaces.

Steklov repeatedly emphasised the importance of the orthogonality condition (20) for the validity of the theory he had developed and adduced appropriated examples. Although mathematical physics had not at that time come across non-selfconjugate problems, Steklov pointed out the interest in them from the point of view of pure analysis and for developing special more general methods of investigation. Here he referred to the works of A. L. Cauchy, J. H. Poincaré, and G. D. Birkhoff. Future development of the theory of non-selfconjugate differential operators was achieved by J. D. Tamarkin, a student of Steklov, and especially by M. V. Keldysh who introduced the important concept of m-multiple completeness of eigenfunctions and their associated functions.

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* Cf. the above cited work of Poincaré, 1894.

** Nowadays this inequality is ascribed to Friedrichs.

5

STEKLOV'S THEORY OF COMPLETENESS

During the thirty years between 1896 and 1926, Steklov published a large number of articles and memoirs in which he developed his *completeness theory*. He did this as he was involved in solving problem B of the justification of Fourier's method that was to elucidate which functions could be expanded into Fourier series in eigenfunctions of the boundary-value problems, X_k, and a more general problem, the expansion of functions in terms of orthogonal systems of functions.

What is Steklov's theory of completeness?

In the case of a finite-dimensional Euclidean space the completeness condition is expressed by *Pythagoras's theorem*. As is well known, every vector a in an n-dimensional Euclidean space, R^n, can be uniquely decomposed into any orthonormal system of n vectors (unit vectors) $e_1, e_2, \ldots, e_n$, $(e_k, e_j) = \delta_{kj}$ according to the formula:

$$a = a_1e_1 + a_2e_2 + \ldots + a_ne_n, \quad a_k = (a, e_k) \qquad (1)$$

Furthermore, the square of the length of the vector is equal to the sum of the squares of those of its components (a_ke_k), i.e.

$$|a|^2 = a_1^2 + \ldots + a_n^2 \text{ (Pythagoras's theorem)} \qquad (2)$$

Here (a, b) is the scalar product of the two vectors a and b from R^n. Identity (2) expresses the completeness condition for the orthonormal system $e_1, \ldots e_n$ in R^n space. Thus we see that every orthonormal system of n vectors is complete in R^n.

Analogous concepts can be introduced for functions if such a function is considered to be an element (vector) of a properly defined linear functional space. However the position is significantly more complicated here. This is related to the fact that the most interesting functional spaces are infinite dimensional. We will consider as an example the set of real measurable functions $f(x)$ given on the interval $(0, l)$ and such that

$$\int_0^l \rho(x) f^2(x)\,dx < \infty$$

where the weight $\rho(x)$ is a continuous positive function on $(0, l)$. The analogue of the length of a vector here is the norm of a function which is defined by the equality:

$$\|f\| = \left[\int_0^l \rho(x) f^2(x)\, dx\right]^{1/2}$$

and the analogue of the scalar product of vectors is the scalar product of functions:

$$(f, g) = \int_0^l \rho(x) f(x) g(x)\, dx$$

The concepts of the norm and scalar product just introduced possess the same properties possessed by the length and scalar product for vectors. This set of functions is a linear set and we call it the space of functions, $L_2^\rho(0, l)$. The space $L_2^\rho(0, l)$ belongs to a category called the *Hilbert spaces* which are the natural generalisations of the *Euclidean spaces* R^n. However, as distinct from a finite-dimensional case we find that in $L_2^\rho(0, l)$ there is an *infinite* (*countable*) *system of orthonormal functions* (for example, system (22) of Sect. 4 is orthonormal in $L_2^\rho(0, l)$ with $\rho = 1$).

Let X_k, $k = 1, 2, \ldots$, be an orthonormal system of functions in $L_2^\rho(0, l)$, i.e. $(X_k, X_i) = \delta_{ki}$. Now the function X_k is playing the role of a unit vector in $L_2^\rho(0, l)$. Every function f in $L_2^\rho(0, l)$ can be associated with the formal series

$$a_1X_1(x) + a_2X_2(x) + \ldots + a_kX_k(x) + \ldots,$$
$$a_k = (f, X_k) \tag{3}$$

which is its (generalised) *Fourier series* in the orthonormal system of functions, $\{X_k\}$, and the a_k are its Fourier coefficients. When does series (3) converge to $f(x)$?

Consider the difference $\varepsilon_n(x)$ between the function and the sum of the first n terms of Fourier series (3), i.e.

$$\varepsilon_n(x) = f(x) - \sum_{k=1}^{n} a_kX_k(x) \tag{4}$$

Using a simple calculation the following identity can be established:

$$\|\varepsilon_n\|^2 = \|f\|^2 - \sum_{k=1}^{n} a_k^2 \tag{5}$$

The number $\|\varepsilon_n\|^2$ is known as the *mean square error* of the approximate representation of $f(x)$ by the partial sum of its Fourier series (3). If $\|\varepsilon_n\| \to 0$ as $n \to \infty$, we say that Fourier series (3) converges to $f(x)$ in the mean on $(0, l)$ or, in short, in $L_2^\rho(0, l)$. It follows directly from identity (5) that: (1) the series composed of squared Fourier coefficients in any orthonormal system always converges, and the inequality

$$\sum_{k=1}^{\infty} a_k^2 \leqslant \|f\|^2, \tag{6}$$

called *Bessel's inequality* always holds true; and (2) for Fourier series (3) to converge to $f(x)$ in $L_2^\rho(0, l)$ it is necessary and sufficient that

$$\sum_{k=1}^{\infty} a_k^2 = \|f\|^2 \tag{7}$$

hold true. The identity is analogous to (2), and is Pythagoras's theorem for a finite-dimensional space.

In connection with the last result, Steklov was the first to introduce the concept of the complete system of functions to mathematics: an orthonormal system $\{X_k\}$ is *complete* if for any function f from $L_2^\rho(0, l)$ identity (7) is true. Steklov called identity (7) the *completeness condition*. Steklov introduced the concept in his note [85] in 1910 although he had been using it practically since 1896 [14, 16, 22, 56].

What has just been considered can be transferred without any essential change to a function of several variables.

A complete system of functions has a property of completeness, i.e. the system cannot be expanded by the addition of new functions without losing orthogonality. In an infinite-dimensional functional space (a Hilbert space) it is obviously impossible to verify the completeness of a system of "unit vectors" by simply enumerating them.

On the other hand, the verification of the completeness condition is a necessary prerequisite to the study of expandability of a function into (uniformly converging) Fourier series by a given orthonormal system of functions. Therefore the proof of the completeness condition for various orthogonal systems of functions becomes very important. The problem of the completeness of orthogonal systems of functions for a number of boundary-value problems of mathematical physics was first raised in its general form by Steklov. Starting in 1896, he was occupied by the topic until the end of his life. His investigations in the theory of completeness and expansion by orthogonal systems of functions were the first in the literature of the world amongst the vast amount of work in the field.

In a memoir of 1904 [59] he collected all the orthogonal systems (11 of them) of functions that were then known and for which he had established the completeness condition (also cf. [53]). These included the classical orthogonal polynomials, the eigenfunctions of the Sturm-Liouville problem (18), (19) of Sect. 4, the eigenfunctions of the boundary-value problem of (40) in Sect. 4, Steklov's fundamental functions (V_k cf. (44) in Sect. 4), and the related functions of Poincaré, Le Roy and Korn for which Steklov established the completeness condition.

The theory of completeness was also considered in [88] (1911), a note [87], and later in notes [99, 100] (1916).

Steklov in his studies of the theory of completeness used a classical concept of integral, namely the Riemann integral. It was later ascertained that the theory took on a complete form if more general concepts of integral were used, i.e. the Lebesgue integral or even the Lebesgue-Stieltjes integral.

The completeness condition subsequently became important for mathematical analysis in general and not just for solving the boundary-value problems of mathematical physics. In particular it made it possible to extend the geometrical concepts of a finite-dimensional Euclidean space to infinite-dimensional Hilbert spaces. Thus, thanks to his theory of completeness, Steklov had come close to the concept of Hilbert space (we would now say he was working in a pre-Hilbert space). Nowadays Steklov's completeness condition (7) is known as Parceval's equality, since he was the first to indicate it without proof for

a trigonometric system of functions (1805). Thus we think it just to call completeness condition (7) the *Parceval-Steklov equality.*

The German mathematician Kneser in his obituary of Steklov printed in 1929 [IX] wrote, "The completeness equation could well be called Steklov's favourite formula and may be called the Steklov formula because he surpassed A. Hurwitz and was the first to prove it rigorously for other cases".

A little later the Soviet mathematician, Academician S. L. Sobolev wrote about these investigations, "Steklov's remarkable ideas have turned out to be very fruitful. His theory of completeness has, without his knowing it, anticipated the concepts and theorems of our modern functional analysis and theory of Hilbert spaces, creating the ground in which these subsequently flourished...."*

It should be noted that the impetus for Steklov's work on the theory of completeness came from the investigations of Lyapunov in 1896 on trigonometric and spherical functions.** The work of Hurwitz***, which was only on the completeness theory of trigonometric functions, appeared in 1903 by which time the basic propositions of the theory were already well known to Lyapunov and Steklov.

Without touching on every method used in his investigations on the completeness theory, we will only show the one Steklov used quite widely since 1907 (cf. [67]), the so-called *method of smoothing*. Let a function $f(x)$ be given on $[0, l]$ and continued outside $[0, l]$ so that $f(x) = f(0)$, $x < 0$ and $f(x) = f(l)$, $x > l$. Now consider the function

$$F(x) = \frac{1}{h} \int_{x}^{x+h} f(\xi)\, d\xi, \tag{8}$$

for all $h > 0$. This is now known as *Steklov's function* and has much better smoothness properties than the original function f, for example, if f is continuous, F is continuously differentiable, and $F(x) \to f(x)$ as $h \to 0$ uniformly. Moreover, if the completeness condition is fulfilled for F, then it will aslo be fulfilled for f.

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* *Mathematics in the USSR over 30 Years*, Moscow, 1948, p. 521.

** Cf. *Proc. Khar. Math. Soc.* for 1896.

*** *Math. Ann.*, 1903, Bd. 57, S. 425-446.

The last result is a particular case of *Steklov theorem* which states that if the completeness condition is fulfilled for a set of functions dense in $L_2^{\varrho}(0, l)$, then it is also fulfilled for every function from $L_2^{\varrho}(0, l)$. Everywhere

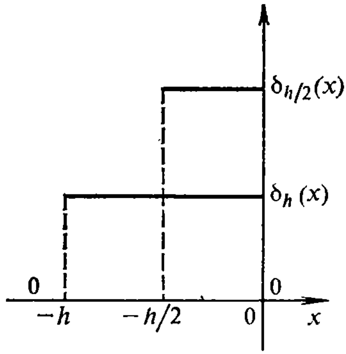

Fig. 3

dense in $L_2^{\varrho}(0, l)$ are the set of polynomials and also the set of functions that are infinitely differentiable with compact support on $(0, l)$.

In modern terms Steklov's function $F(x)$ is none other than the convolution of the original function $f(x)$ and the function

$$\delta_h(x) = \begin{cases} \frac{1}{h} & -h \leqslant x \leqslant 0 \\ 0 & x < -h \text{ or } x > 0 \end{cases}$$

The sequence of functions, $\delta_h(x)$, as $h \to 0$, is a δ-like sequence (Fig. 3), i.e. $\delta_h(x)$ weakly approaches the Dirac δ-functions*, so that $F(x) = f(x) * \delta_h \to f(x)$ as $h \to 0$ uniformly.

In this way Steklov's method of smoothing is none other than the method of (averaging) regularisation of (generalised) functions, a method widely used in modern mathematics.

Steklov, in [22, 34, 44, 53, 63] gave sufficient conditions for the expansion of functions of several variables into

* An elementary introduction to the theory of generalised functions can be found in the book by V. S. Vladimirov (Ch. II) cited above.

uniformly converging Fourier series in orthogonal systems of functions for a number of boundary-value problems of mathematical physics. Thus for the eigenfunctions of problem (40) in Sect. 4 he established that every twice continuously differentiable function which satisfies the associated boundary condition for a domain bounded by Lyapunov surfaces could be expanded into an absolutely and uniformly converging Fourier series.

A great deal of Steklov's work is concerned with the expandability into the eignefunctions of the Sturm-Liouville problem ((18)-(19) of Sect. 4) for different assumptions about the coefficients ρ, p and q and for more general boundary conditions which ensure the orthogonality of the eigenfunctions on $(0, l)$ with weight ρ (cf. Eq. (20) of Sect. 4) [16, 46, 59, 68, 69, 79-91]. He gave a systematic presentation of all the results pertaining to here in the monograph *Basic Problems of Mathematical Physics* [118].

By improving and developing the Schwarz-Poincaré method and by using his own theory of completeness, Steklov started in 1896 gradually to establish all the more exact theorems on the expansion into eigenfunctions of the Strum-Liouville problem. It is not possible in these short sketches to trace through all the stages of this large work of his (this was done by N. M. Gyunter in appendix III of [VIII]).

We draw attention to the fact that Steklov's work [63] (1904) is a response to Hilbert's remarkable communication on the theory of integral equations with a symmetric kernel. Steklov used his methods to establish theorems about expansion that were fully analogous to Hilbert's theorems. Soon after the dissertation by E. Schmidt (1905) the Hilbert-Schmidt theory combined both approaches. However, the Hilbert-Schmidt theorem, which states that a function source-wise representable in terms of a symmetric kernel can be expanded into eigenfunctions of this kernel, requires excessive smoothness of the function in question.

By way of example we cite the well-known result of Steklov of 1907 [68] which does not follow from the Hilbert-Schmidt theorem.

If a function $f(x)$ is continuously differentiable on $[0, l]$ and satisfies the boundary condition $f(0) = f(l) = 0$ then it can be expanded in a uniformly converging Fourier series

in the eigenfunctions of the Sturm-Liouville problem ((18), (19) of Sect. 4).

In later work of 1910 [81, 84] and in a memoir of 1913 [91] Steklov extended his results to even wider classes of functions and for even more general boundary conditions (also cf. [118]).

As regards the methods Steklov used to prove the expansion theorems, it is necessary to note the following important result of his, viz., if an orthonormal system $\{X_k\}$ is complete, $f \in L_2^\rho\ (0,\ l)$, and series (3) is a Fourier series of the function f, then for any ϕ from $L_2^\rho\ (0,\ l)$ the following identity is true:

$$\int_0^l \rho f \phi\, dx = \sum_{k=1}^{\infty} a_k \int_0^l \rho X_k \phi\, dx \tag{9}$$

This result, besides being applied to problems of expansion, is important in principle. It shows that even if the formal Fourier series (3) diverges or does not represent the function $f\ (x)$ at the points x, then it can still be integrated term-by-term with any function ϕ from $L_2^\rho\ (0,\ l)$ and the number series obtained will converge to the value $(f,\ \phi)$. Thus series (3), as we now say, *weakly converges* in $L_2^\rho\ (0,\ l)$. This fruitful idea of first *averaging* a formal series or some other formal mathematical object (a diverging integral, a derivative of a non-differentiable function, etc.) and then relating to this object finite values (the results of the averaging) later led to the concept of a *function of a set* and then to the concept of a *generalised function*, a notion basic to modern mathematics. These concepts are now firmly entrenched in mathematics, physics, and engineering. They have drawn together mathematical physics and the real conditions of physical experiments.

We now just draw attention to Steklov's work on the expansion of functions in the classical orthogonal polynomials, i.e. Jacobian polynomials [54, 57, 106-108, 115], Hermitian polynomials [101, 106-108], Laguerre's polynomials [102, 103], and orthogonal polynomials with non-negative weights which may only vanish at distinct points* [54, 59, 69, 94, 103, 106-108, 113-115, 121, 125-126]. Steklov used the theorems he had obtained for the

* Steklov called these Chebyshev polynomials [54].

expansion in eigenfunctions to prove the existence and uniqueness of the solution of the boundary-value problems given by Eqs. (12), (13), (14) and (28), (29) of Sect. 4. He also showed the sufficient conditions for the functions ϕ_0 and ϕ_1 which are required for these solutions to be represented in terms of Fourier series (25) and (30) (cf. Sect. 4), respectively.

6

HIS WORK IN MECHANICS

V. A. Steklov began his research and educational activities in the field of mechanics. The first such work appeared in print in 1891 and his last appeared in 1909. He obtained some important results in this field. He wrote 17 works on hydrodynamics, four on the theory of elasticity, and three on analytical mechanics.

In hydrodynamics Steklov's name is connected above all to one of the few integrable cases of the motion of a rigid body in liquids. He also studied the theory of vorticity, the motion of a liquid ellipsoid, and the motion of a rigid body with an ellipsoidal cavity filled with liquid.

When Steklov's work on the motion of rigid bodies in a liquid appeared, the topic had been studied by many leading scientists. Even Bernoulli and Euler had studied it during the middle of the 18th century, it had also been investigated by Poisson at the beginning of the 19th century, and Stokes's work had appeared in the 1840's. Stokes showed that the pressure of a liquid on a moving body played as important a role as an increasing mass of the body. He was the first to attract attention to the problem of the motion of bodies with cavities filled with liquid and showed that the liquid could be replaced by an equivalent rigid body. In 1852 Dirichlet obtained the motion of a sphere in an ideal liquid and showed the way for obtaining the general equations of motion of a rigid body in a liquid. In 1856 Clebsch considered a more general case, the motion of an ellipsoid in a liquid, and in 1867, V. Thomson and P. Thet showed another way to derive the equations of motion for a rigid body in a liquid, viz., using Hamilton's principle. Then in 1870, G. Kirchhoff developed the idea and represented the equations of

motion of a body in a liquid in the form the equations have for the motion of a body in a vacuum. He obtained three integrals of motion of a body and then reduced to quadrature the solution of the motion of a body of revolution in a liquid.

Later work on the topic was carried out in the 1870's by Clebsch, Kirchhoff, Thomson, Neumann, Lamb, and others. The work of the Russian scientists Zhukovskii (1847-1921), Lyapunov, Steklov, and Chaplygin (1869-1942) on the motions of bodies in liquids appeared in the 1880's and 1890's.

Starting in 1891, Steklov published his work on hydrodynamics in both Russian and foreign journals. He summed up his earlier work [2, 3, 8, 10] in his master's thesis *On the Motion of a Rigid Body in a Liquid* [9] which appeared in 1893. In this work Steklov derived the differential equations of motion of a body in a liquid for very general assumptions: (1) the body was bounded by a closed multiply connected surface and had one or more inner cavities filled with the liquid; (2) the liquid was ideal, incompressible, and filled all the space outside the body; (3) the velocity of points of the liquid both inside the cavities and outside the body had a potential, the velocity of the liquid at infinity was zero; (4) the forces acting on the body were arbitrary, but those acting on the liquid were conservative.

In a later presentation Steklov did not pay any attention to the cavities having in mind the idea that the liquid inside a cavity could be treated, in the case of a simply connected cavity, like an equivalent rigid body (as Stokes had shown) and in the case of a multiply connected cavity, like an equivalent rigid body and a gyroscope.

Steklov used the method of successive approximation to integrate his differential equations on condition that the ratio of the density of the liquid surrounding the body to the mass of the body and the liquid inside the cavities was sufficiently low, that the motion was due to inertia, and that the external surface of the body was simply connected. Later the author described the various possible motions of the rigid body in the liquid. Finally, he found a new case of integrability of the differential equations of motion of a rigid body in a liquid which had not earlier been noted by Clebsch. His results promoted Lyapunov to find yet

another, final case where the equations permitted a fourth homogeneous, second-degree integral.

Lyapunov gave Steklov's master's thesis [9] an interesting appraisal in his review [LII].

S. A. Chaplygin, also a young scientist at the time, wrote of Steklov's work, "... it includes a review of all the relevant material obtained by various scientists up to its appearance. Therefore he gave his own detailed derivation of the equations of motion without constraining the surface of the body to being simply connected, he showed ... new cases of integrability, described several types of motion of a heavy body in a liquid, and considered various particular cases of solving the problem without the action of forces..."*

In [12] Steklov considered a rigid body moving in a liquid and symmetrical with respect to the three mutually perpendicular planes. He also considered a body of a more general type whose kinetic energy expression not only contained the squares of the projections of the total vector and the total moment of momentum, but also contained three terms with products of the respective projections of the vector and the moment. He showed the case when three particular linear integrals of motion exist simultaneously.

Steklov later developed the method of successive approximations for integrating the differential equations of motion of a rigid body in a liquid in [52], the method he had presented in his master's thesis. As a result of his investigation he reduced the solution of the above problem to an integration of the differential equations of motion of a free rigid body in a vacuum and to the integration of certain linear differential equations with variable coefficients. Using successive approximation, he proved that the differential equations of motion of a rigid body in a liquid (the body was bounded from outside by a simply connected surface) had an infinite number of periodic solutions, given the ratio of the density of the surrounding liquid to the mass of the body and the liquid in cavities was low. The total period was equal to the period of the solutions when the ratio was zero.

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* S. A. Chaplygin, *Math. Collection*, Vol. XX, No. 1-4, Moscow, 1897.

In [15] Steklov studied the motion of a viscous incompressible liquid that asymptotically approaches the rest state assuming that the projections of the vortex at some point are proportional to the projections of the velocity at that point, and consequently the vortex line coincided with the lines of flow, the proportionality coefficient being constant. He proved using successive approximation that the motion exists and is defined if the normal component of the velocity was known at the surface that bounded the liquid, the component being expressed as a product of the coordinate functions by the function $e^{-\alpha t}$ where $\alpha > 0$ and t the time.

In [71] Steklov gave a general solution of the problem of the motion of an incompressible liquid, viz. given the vortices in the liquid contained in a closed simply connected vessel which is moving according to a known law, to determine the velocities of the liquid points. He showed that in order to solve the problem it is sufficient to integrate the system of equations, which is made up of the continuity equations and the three differential equations defining the vortex as a function of time and the coordinates of the point, provided that the normal components of the velocity of any point in the liquid next to the vessel surfaces and the velocity of the corresponding point of this surface are equal to each other. Steklov reduced the problem to solving Neumann's problem and as a result obtained expressions for the projections of the velocity that contained two arbitrary functions, a suitable choice of which simplified the procedure in concrete cases. We note that until Steklov's investigations only the following two particular cases had known solutions of this problem: (1) an unbounded liquid at rest at infinity; (2) a liquid filling a stationary vessel, the velocity of the liquid points next to the vessel surface being known. Steklov considered a number of particular cases for finding the velocity of the liquid for the vortex in [71, 72, 77].

Another large topic in hydrodynamics that Steklov dealt with was related to the motion of a liquid mass maintaining an ellipsoidal form, the particles of which are mutually attracted according to Newton's law. The topic had attracted the attention of many leading mathematicians because it was closely related to one of the most

important problems of celestial mechanics, namely 'what form did celestial bodies have?'

Some fundamental memoirs by Dirichlet and Riemann appeared after the works of Maclaurin, Laplace, and Jacobi that had been devoted to ellipsoidal forms of equilibrium of rotating liquids. Dirichlet and Riemann considered the various cases of motion of a liquid (homogeneous, ideal, and incompressible) maintaining an ellipsoidal form. It was assumed that at every moment of time the coordinates of every point of the liquid were linear homogeneous functions of their initial coordinates and that the centre of the ellipsoid could be considered to be stationary (Dirichlet's assumption).

Steklov dealt with the problem in [64-66] (1905-1906) and [72] (1908-1909). In these he studied all the possible motions of a liquid ellipsoid using Dirichlet's assumption and also assumed the ellipsoid was either an ellipsoid of revolution or triaxial with the axis invariable in length. He proved that these forms had only three kinds of motion (a Dirichlet case and two Riemann cases) which had been investigated in depth in the above mentioned works.

Steklov's work [77] was very important for astronomy and celestial mechanics. In it he studied the rotation of a rigid body about a fixed point, the body having an ellipsoidal cavity filled with an incompressible liquid, the axes of the ellipsoid coinciding with the principal axes of inertia of the body relative to the fixed point. It was assumed that only the gravitational force of mutual attraction acted on the particles of the liquid and that the vortex filaments were straight lines with equal stress. By using the equations in [71] and assuming the moment of the external forces was zero, Steklov obtained three integrals of motion for these equations. By considering the case when the mass of the liquid was quite small compared to the mass of the body, he could integrate the system of differential equations of motion he had compiled using successive approximation and showed periodic motions existed. By considering particular cases he obtained different types of motion. The case where the body and its cavity were ellipsoids of revolution about a common axis can be integrated completely in elliptical functions. Finally, Steklov showed that on going over to the general case of a viscous liquid and

assuming that the moment of the external forces relative to the fixed point was zero, the motion of the system tended to a uniform rotation of a solid about one of the principal axes of inertia.

The results he obtained enabled Steklov to come to some conclusions about the nature of motions of the poles of the Earth and the changes in its latitudes. He could also estimate the density and thickness of the Earth's crust and also the density of the liquid mass.

Steklov also obtained some other interesting results in hydrodynamics but it is not possible to mention them here (e.g. see [47]). As I. V. Meshcherskii has said [VIII], "the name Vladimir Andreevich Steklov will be remembered forever in hydrodynamics".

Steklov published four works on the theory of elasticity which date to the first years (1891-1898) of his independent research.

In his work [4] he studied the equilibrium of an infinitely thin twisted rod whose cross section was such that the two moments of inertia about the two principal axes are equal to each other, i.e. a circle or regular polygon. It was assumed that the rod was acted on by external forces applied to its ends and that there was a constant pressure on the line of the centre of gravity of the cross sections and along the principal normal to the curve. Kirchhoff had solved in general the problem on the equilibrium of a thin rod when one or two of the dimensions of the cross section were very small compared to the length of the rod. He had derived the appropriate differential equations for arbitrarily located forces but had only integrated them for the case where the forces were applied to the ends of the rod. Steklov started from Kirchhoff's basic equations and applied them using Clebsch's presentation.

Using the assumptions given above, Steklov integrated the differential equations for the equilibrium of an elastic rod expressing the elements that characterised the rod's deformation in elliptic integrals and Jacobian θ-functions. Finally, Steklov showed how using the equations he had obtained to go over to the planar equilibrium case shown by M. Levi. Steklov's paper [5] dealt with the Saint-Venants problem of the equilibrium of a cylindrical or prismatic elastic rod, assuming that one of its ends was

fixed and external forces were applied to the other end whilst the lateral surface and internal mass were free from the effects of the forces. We consider that the z-axis was directed along the cylinder element and the origin was at one of its cross sections. Saint-Venants started from the following physical hypothesis: the stress components X_x, X_y, Y_x, Y_y are zero at every point inside the cylinder. He found the deformation and stress in the cylinder given that the stress at the lateral surface was zero.

When he studied the problem, Steklov made no assumptions about the stress and showed that if there were no forces acting on the lateral surface of the cylinder and on its internal mass, then there was a unique general solution (when any straight line parallel to the cylinder's axis can be transformed into an algebraic curve) was the Saint-Venants solution (i.e. can be transformed into a third order algebraic curve). Steklov studied in the same work another case where the forces act on the lateral surface too.

In [28] Steklov returned to the problem of equilibrium of elastic isotropic cylindrical bodies. Giving the values of the displacement as

$$u = u_0 + u_1 z + u_2 \frac{z^2}{1\cdot 2} + u_3 \frac{z^3}{1\cdot 2\cdot 3} + \ldots$$

(analogous expressions for v, w, and θ), where the coefficients of z were functions of x and y, he obtained a solution which combined, in a mathematical sense, the solutions of the Saint-Venants and Clebsch problems and was more general. Clebsch's problem was to find the equilibrium conditions for a plate or a cylinder whose height was small compared to its cross section, and where external forces acted on its lateral surface. He also considered the X_z, Y_z and Z_z stresses to be zero.

In [7] Steklov considered the equilibrium problem of elastic bodies of revolution. He applied his general conclusions to the study of some separate cases: (1) the equilibrium of a circular cylinder under the action of forces applied to its lateral surface; (2) the equilibrium of a hollow cylinder under the action of a force applied to its base; (3) the equilibrium of bodies of revolution bounded by spheres and conic surfaces.

We shall now consider Steklov's three works in analytical mechanics. In this area he studied the classical problem of the motion of a rigid body about a fixed point in a vacuum. This topic had occupied Euler, Lagrange, Poisson, Kovalevskaya, and Zhukovskii.

In two articles, [13] and [29], Steklov found some new special cases of the integrability of the equations of motion of a heavy body which had a fixed point. One of these cases was being studied at the same time by D. K. Bobylev and so entered the literature as the *Bobylev-Steklov* case.

Steklov devoted [19] to a presentation of a transformation, now known as the *Steklov transformation*, and its application to various problems of mechanics. He brought out many examples to illustrate the application of his method. In particular, he showed that using the method he could obtain the results D. N. Goryachov had discovered in 1895 using another method in *On the Three Body Problem.*

All of Steklov's work on mechanics was distinguished by the generality of his questions and his rigorous mathematical treatment. His work yielded both new conclusions and the development of results obtained earlier, introducing essential additions to the investigations of authorities such as Dirichlet, Riemann, Clebsch, Saint-Venants, Kirchhoff, Thomson, Kovalevskaya, Zhukovskii, and Lyapunov.

7

QUADRATURE FORMULAE *

Let $\rho(x)$ be a given integrable function on the interval $(-1, 1)$. A *quadrature formula*** with a weight ρ and n nodes is given by the approximation formula

$$\int_{-1}^{1} \rho(x) f(x)\, dx \approx \sum_{k=1}^{n} A_k^{(n)} f(x_n^{(n)}) \tag{1}$$

* Steklov called quadrature formulae mechanical quadrature formulae.

** The analogous formulae for functions of several variables are known as cubature formulae.

The $A_k^{(n)}$ and nodes $x_k^{(n)}$ are chosen so that approximation (1) turns into an equality for all polynomials of degree p_n. The number p_n is known as the *degree of accuracy* and the expression

$$R_n[f] = \int_{-1}^{1} \rho(x) f(x)\, dx - \sum_{k=1}^{n} A_k^{(n)} f(x_k^{(n)}) \qquad (2)$$

is called the *remainder term* of quadrature formula (1). We will assume from now on that $p_n \to \infty$ as $n \to \infty$.

In particular, R. Cotes has suggested, for $\rho(x) \equiv 1$, choosing equidistant nodes that give a quadrature formula with a degree of accuracy $p_n = n - 1$. Gauss in trying to obtain greater accuracy suggested a quadrature formula in which the nodes were the roots of an appropriate Legendre polynomial which gave a degree of accuracy $p_n = 2n - 1$, the highest possible. Chebyshev tried to simplify the calculation and suggested a choice of nodes where the coefficients would be the same so that $p_n = n$* for even n, and $p_n = n + 1$ for odd n. Markov suggested fixing the end nodes $x_1^{(n)} = -1$ and $x_n^{(n)} = 1$ and choosing the remaining nodes and coefficients so that the degree of accuracy was $2n - 3$. Three basic problems arose due to quadrature formulae:

(1) The convergence of the quadrature formula, i.e. indicating the sufficient conditions for a function f so that the remainder term $R_n[f] \to 0$ as $n \to \infty$.

(2) Estimating the remainder term of the formula for a given function (or class of functions).

(3) Choosing the best practical quadrature formula for a given class of functions.

Before Steklov these questions had been tackled in a number of works by Gauss, Cotes, Chebyshev, Sonin, Posset, Stieltjes, Peano, and Markov. In a series of works published in the *Proceedings of the Academy of Sciences* from 1916 to 1919, Steklov achieved some outstanding results of great generality in all the indicated directions and he frequently applied his theory of completeness and method of smoothing.

Steklov devoted a remarkable article, [98], to the convergence of quadrature formulae in 1916. Using an easily

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* Chebyshev's formula does not exist for $n = 8$ or $n \geqslant 10$. The nodes are complex when $\rho = 1$.

provable formula, viz.

$$R_n[f] = \int_{-1}^{1} \rho(x)\, q_{p+1}(x)\, dx - \sum_{k=1}^{n} A_k^{(n)} q_{p+1}(x_k^{(n)}) \tag{3}$$

in which $q_{p+1} = f - P_p$ and $P_p(x)$ is any polynomial of degree $p \leqslant p_n$, he reduced the problem to the problem of approximating the function $f(x)$ by polynomials. Assuming $P_p(x)$ to be the sum of the first p terms of the expansion of $f(x)$ in Legendre polynomials, Steklov established the following estimate:

$$|R_n[f]| \leqslant \frac{1}{2\sqrt{p_n}} \left(\sum_{k=1}^{n} |A_k^{(n)}| \right.$$

$$\left. + \int_{-1}^{1} |\rho(x)|\, dx \right) \max |f''(x)| \tag{4}$$

This estimate shows that every quadrature formula for which

$$\sum_{k=1}^{n} |A_k^{(n)}| \leqslant K \tag{5}$$

will converge for all functions with a bounded second derivative. Furthermore, by applying the method of smoothing twice, he proved the convergence of these formulae for all continuous functions. This rather general result of his was a generalisation of the well-known theorem of Stieltjes on the convergence of Gauss's quadrature formula, and is now a classic.

Quadrature formulae of the Gauss and Chebyshev type satisfy equation (5) since $A_k^{(n)} > 0$ for them. Thus these formulae will always converge for all continuous functions.

Some quadrature formulae, particularly Cotes's and Markov's formulae, do not satisfy (5). Thus Steklov considered a case for which condition (5) could be changed to the more general

$$\sum_{k=1}^{n} |A_k^{(n)}| \leqslant K n^{\mu}, \quad \mu > 0 \tag{6}$$

Markov's quadrature formula satisfies this condition for $\mu = 1$. Steklov proved convergence of these quadrature

formulae for functions $f(x)$ such that $f^{(s)}(x)$ satisfies a Lipschitz condition where s is an integer that obeys the inequality

$$s > \mu - \frac{1}{2}$$

Cotes's quadrature formula does not satisfy condition (6) but does satisfy the more general one

$$\sum_{k=1}^{n} |A_k^{(n)}| \leqslant K \frac{2^n}{n^{3/2} \ln^2 n} \tag{7}$$

Steklov proved that Cotes's quadrature formula converged for functions $f(x)$ that were analytic in some neighbourhood of the segment $[-1, 1]$. Finally he proved the convergence of quadrature formulae with positive coefficients for all Riemann integrable functions.

Steklov devoted a second remarkable article [104] and a considerable part of his wide ranging works [106-108] to the estimation of the remainder terms of quadrature formulae. The starting point of his investigation was the approximation of a function using orthogonal polynomials with weight ρ (Chebyshev polynomials*). Applying recurrence relations to these polynomials, Steklov obtained an expression for the remainder term (error) when the function f is expanded into Chebyshev (normalised) polynomials, $\mathcal{P}_p$ (with weight ρ) in the form of the integral:

$$\rho_p(x) = \int_{-1}^{1} \rho(y) \frac{f(x) - f(y)}{x - y} [\mathcal{P}_{p+1}(x) \mathcal{P}_p(y) - \mathcal{P}_p(x) \mathcal{P}_{p+1}(y)]\, dy$$

By skilfully applying an elementary identity he had derived in [105], which was none other than the formula of a multiple integration by parts, he transformed the expression of the remainder term to a form which contained higher order derivatives of the function f. Thus Steklov obtained, taking Legendre polynomials as an example, the remainder term in the form

$$\rho_p(x) = \frac{2}{(2p+1)!!} \left[f^{(p+1)}(\xi) - \frac{M+m}{2} \right] \tag{8}$$

* Cf. footnote on p. 74.

where ξ is some number from the interval $(-1, 1)$, and M and m are the greatest and least values of $f^{(p+1)}(x)$ on $[-1, 1]$ respectively.

He obtained an accurate estimate which can be seen from the fact that for expansions in Chebyshev polynomials $T_n(x)$ (corresponding to the weights $\rho(x) = (1-x^2)^{-1/2}$), we get:

$$|\rho_p(x)| \leqslant \frac{\sigma_p}{2^p(p+1)!} \max_{-1 \leqslant x \leqslant 1} |f^{(p+1)}(x)| \tag{9}$$

where $\sigma_p \to 1$ as $p \to \infty$ and $1 < \sigma_p < 1.52$.

Starting from Eq. (8), Steklov obtained an expression for the remainder term of the quadrature formula in the form

$$R_n[f] = \frac{2H_n}{(2p_n+1)!!}\left[f^{(p_n+1)}(\xi) - \frac{M+m}{2}\right] \tag{10}$$

where

$$H_n = \int_{-1}^{1} |\rho(x)|\,dx + \sum_{k=1}^{n} |A_k^{(n)}|$$

He used the general formula (10) to obtain rather simple expressions and estimates for the remainder terms of the Cotes and Chebyshev quadrature formulae. Steklov also obtained estimates for the remainder terms of the Gauss and Markov quadrature formulae and compared them to the estimates that were known earlier. Interestingly, his estimates, which were obtained from general considerations, did not much exceed the earlier known estimates, which had been obtained using particular features of the Gauss and Markov quadrature formulae. For example, with $n = 5$ in Gauss's formula the ratio between the estimates does not exceed 1.4.

Finally, Steklov showed an estimate of the remainder term in the form

$$|R_n[f]| \leqslant H_n E_{p_n}(f) \tag{11}$$

where $E_{p_n}(f)$ is the best approximation to the function f by polynomials of degree p_n. Estimates known at that time of $E_p(f)$ gave worse estimates of the remainder term than did the estimate of $R_n[f]$ in (10), and Steklov posed the problem of finding a method to calculate accurately

the best approximation $E_{p_n}(f)$. This topic was later developed very widely and recently a whole branch of mathematics, the constructive theory of functions, has grown out of it.

Steklov devoted articles [109-112] to the search for the most practical quadrature formulae. The effort to improve the Cotes and Chebyshev quadrature formulae led him to believe that it was advantageous in complex formulae to sacrifice the simplicity of the node distribution on one hand while on the other it was desirable to have simple (rational) expressions for the coefficients, with a condition that the maximum degree of accuracy is preserved. Assuming that the odd moments of all orders, less than or equal to p_n, of the weight function ρ are zero, Steklov studied the most important cases of quadrature formulae with small $n = 3, 4, 5, 6, 7$. For each of these cases Steklov showed the most advantageous quadrature formulae as regards simplicity of calculation and accuracy. In particular for $n = 5$ he obtained the interesting quadrature formula

$$\int_{-1}^{1} f(x)\,dx \approx \frac{32}{45} f(0) + \frac{49}{90}\left[f\left(-\sqrt{\frac{3}{7}}\right) + f\left(\sqrt{\frac{3}{7}}\right)\right] + \frac{1}{10}[f(-1) + f(1)] \tag{12}$$

which has a remainder term

$$R_5[f] = -\frac{2^5}{5\cdot 3^2\cdot 7^2}\,\frac{f^{(8)}(\xi)}{8!} = Af^{(8)}(\xi)$$

Interestingly, Gauss's quadrature formula with $n = 4$ also has the same degree of accuracy with its remainder term being equal to $0.8Af^{(8)}(\xi')$.

It is appropriate to mention in this short review of Steklov's work on quadrature formulae, the well-known formula he obtained in 1913 in [93]; viz.

$$f(x+0) - f(x+h-0) = \frac{4}{h}\sum_{k=0}^{\infty}\int_{x}^{+h} f(\xi)\cos\frac{(2k+1)\pi(\xi - x)}{h}\,d\xi \tag{13}$$

This is true for any function $f(x)$ of restricted variation. It has many applications, in particular, the theorem about the expansion of functions in trigonometric series follows from the formula, as do the asymptotic formulae that are related to the Γ-function, and the properties of Riemann's ζ-function.

We should note, finally, that Steklov had a great mastery of the mathematical apparatus of analysis and obtained many and unexpected corollaries from comparatively simple formulae. We should also mention that besides the work cited [93], he also published [55, 56, 58, 59, 105] on the subject.

8

ASYMPTOTIC METHODS

Asymptotic methods in analysis had already been conceived in the 18th century and widely applied by Lagrange, Laplace, and Leverrier who laid the foundations of the theory of perturbations.

Asymptotic methods are in general applied in the theory of ordinary differential equations to the following types of problems:

(1) the behaviour of the solutions of differential equations when an independent variable tends to a limit;

(2) the behaviour of the solutions of differential equations when a parameter in the equation or the supplementary conditions tends to a limit;

(3) the behaviour of solutions when the independent variable and a parameter tend to limits simultaneously;

(4) determining the solutions of a differential equation which have a given asymptotic behaviour (are bounded or stable, or have a certain growth, or are periodic or nearly periodic).

Asymptotic methods for solving ordinary differential equations with a large parameter have their beginnings in the work of Sturm and Liouville. They applied them to an investigation of the expansions of a self-conjugate boundary-value problem into eigenfunctions. The methods arise from the attempts to justify Fourier's method (separation of variables, cf. Sect. 4). In 1837 Liouville was the first to establish the structure of the fundamental system of solu-

tions for a singularly perturbed equation of the form

$$y''(x) + [\lambda^2 r(x) + g(x)]\, y(x) = 0$$

when $\lambda \to \infty$, $r(x) > 0$ and $g(x) \geqslant 0$. He showed that as $\lambda \to \infty$ either of the two linearly independent solutions of the equation could be formally rendered in the form of series*:

$$y_i(x, \lambda) = \left[y_{0i}(x) + \frac{1}{\lambda} y_{1i}(x) + \ldots \right]$$

$$\times \begin{cases} \sin\left[\lambda \int\limits_0^x \sqrt{r(x)}\, dx\right] \\ \cos\left[\lambda \int\limits_0^x \sqrt{r(x)}\, dx\right] \end{cases}$$

$$i = 1, 2;\ x \in [a, b]$$

In fact this was an expansion into asymptotic series but at that time the concept did not exist as it was first introduced by Poincaré**. The first serious mathematical justification of these methods was achieved by Poincaré in his large opus *New Methods of Celestial Mechanics* (1892). Lyapunov and Steklov developed asymptotic methods for the theory of self-conjugate ordinary differential equations with a parameter.

L. Schlezinger*** in 1907 and G. Birkhoff**** in 1908 proved a theorem about the structure of the fundamental system of solutions (the first along a ray and the second in a sector) for a general (non-self-conjugate) singularly

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* The multiplier is sine for one solution and cosine for the other.

** *Acta Math.*, 8, 1886, pp. 295-344. We now say that the function $f(x, \lambda)$ can be represented as an asymptotic series $f(x, \lambda) = \sum\limits_{n=0}^{\infty} C_n(x)\lambda^{-n}$ if for every natural number m there is a constant K_m such that $\left| f(x, \lambda) - \sum\limits_{n=0}^{m} C_n(x)\lambda^{-n} \right| \leqslant \frac{K_m}{|\lambda|^{m+1}}$ for all sufficiently large λ and $x \in [a, b]$.

*** *Math. Ann.*, 1907, Bd. 63, S. 27-30.

**** *Trans. Amer. Math. Soc.*, 1908, V. 9, pp. 219-231.

perturbed ordinary linear differential equation of an nth order with a large parameter.

We will now dwell briefly on Steklov's investigations of asymptotic methods in the theory of ordinary differential equations. These methods arise in connection with problem (C) of the justification of Fourier's method (cf. Sect. 4) when the asymptotic behaviour of the eigenvalues and eigenfunctions of linear differential operators must be studied.

He had this problem in mind in his work on the topic when he provided a general method for obtaining asymptotic expressions for solving linear self-conjugate differential equations. Steklov used these expressions to find the sufficient conditions for expanding functions in Fourier series in eigenfunctions. The results chiefly refer to second- and fourth-order ordinary linear differential equations which contain a large parameter. In addition he investigated Jacobi polynomials [54, 57, 106-108, 115], Hermitian polynomials [101, 106-108], Laguerre polynomials [102, 103], and Chebyshev polynomials* [54, 59, 69, 94, 106-108, 113-115, 121, 125, 126] and constructed asymptotic formulae for them in an independent variable.

The methods Steklov studied are based on the reduction of

$$y'' + [\lambda^2 - q(x)]\, y = 0$$

(itself a reduction of a general second-order self-conjugate differential equation by the substitution of the independent variable and the function) to Volterra's integral equation and the application to this equation of the method of successive approximation [69]. These methods go back to Liouville; however neither Liouville nor any other author that used them (e.g. Kneser) applied them to systems of functions that are as general and important to analysis as the classical orthogonal polynomials. Meanwhile, these polynomials are related to boundary-value problems that have singularities ($q(x)$ becomes infinite in the case of Jacobi polynomials or when an infinite interval exists for the Hermitian and Laguerre polynomials) and so do not belong to the usual Sturm-Liouville scheme. The first person to do this was Steklov who also found

* Cf. footnote on page 74.

that by using some very simple and elegant considerations he could give a convenient estimate for the remainder terms of the asymptotic expressions he had obtained. Modern literature calls the method he developed to obtain the asymptotic expressions for classical orthogonal polynomials the *Liouville-Steklov method.* An application of this method enabled Steklov to prove theorems on the expansion in eigenfunctions of the Sturm-Liouville problem with the same degree of generality that could be done for ordinary trigonometric functions, while placing fewer restrictions on the equations' coefficients than were in Kneser's work*. Steklov elucidated in [69] the conditions under which the convergence of a generalised Fourier series for a function $f(x)$ is equivalent to the convergence of an "approximate" series obtainable by replacing the functions in terms of which the expansion is performed by the first terms of their asymptotic expressions.

Since the first term in the asymptotic expressions obtained by Steklov always had the form

$$a_n\psi(x)\sin\lambda_n\phi(x) \quad \text{or} \quad a_n\psi(x)\cos\lambda_n\phi(x)$$

the theorems he obtained established the simultaneous convergence of a generalised Fourier series and a trigonometric series for certain classes of functions.

From this point of view Steklov's theorems are very similar to the so-called equiconvergence theorems that A. Haar and G. Szegö subsequently proved in the most general form. In note [83] (1910) Steklov came very close to the theorem on equiconvergence and actually proved it for every point where $f(x \pm 0)$ is meaningful. Relying on his theory of completeness Steklov proved that the sum of a generalised Fourier series for the function $f(x)$ coincides with the function (or $\frac{1}{2}[f(x+0)-f(x-0)]$ at discontinuities). He needed to use the second terms of the asymptotic formulae to do this. In the same note Steklov obtained, in essence, the same result on equiconvergence that E. Hobson did in 1906.

Thus we must consider Steklov to be the first mathematician not only to prove a number of important theorems on completeness for various orthogonal systems of func-

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* *Math. Ann.*, Bd. 58, 1904, S. 81-147; Bd. 60, 1905, S. 402-423; Bd. 63, 1907, S. 477-524.

tions, particularly polynomials, but also to lay the foundations for the spectral theory of self-adjoint operators. He realised the importance of his results for analysis and its applications.

Interestingly, the series Steklov obtained had the form

$$u(x, \lambda) = N(\lambda)\left[\psi_0(x, \lambda) + \frac{\psi_1(x, \lambda)}{\lambda} + \frac{\psi_2(x, \lambda)}{\lambda^2} + \dots\right]$$

where λ runs over an unlimited growing sequence λ_1, λ_2, λ_3, . . . and each function $\psi_k(x, \lambda)$ is bounded for all these λ and at all x belonging to a fixed segment in the interval (a, b). He obtained an estimate for the remainder term

$$R_k(x, \lambda) = O\left(\frac{1}{\lambda_k}\right)$$

This enabled him to use this series to obtain various asymptotic representations of the solution $u(x, \lambda)$ for large λ.

It should be emphasised that the asymptotic series Steklov obtained were different from those of Poincaré. The latter had the form:

$$u(x, \lambda) = N(\lambda)\left[a_0(x) + \frac{a_1(x)}{\lambda} + \frac{a_2(x)}{\lambda^2} + \dots\right]$$

where the coefficients $a_k(x)$ are difined uniquely, if in fact they can be defined. By contrast in Steklov's series the choice of functions $\psi_k(x, \lambda)$ has an arbitrariness which makes it possible to restrict them to certain conditions.

Furthermore, note that the first terms of Steklov's asymptotic formulae are expressed via trigonometric functions. The reason for this is that in order to obtain Volterra's integral equation (we mentioned it earlier) on the left-hand side of the differential equation he had to use an operator like

$$L = \frac{d^2}{dx^2} + \lambda_m^2$$

But if there is another, quite well studied, operator on the left-hand side of the differential equation then it is possible to get other formulae. For example, if we take the Bessel operator in place of the one indicated above, then we get a Hilba-type formula for the classical orthogonal polynomials.

Steklov later obtained analogous results for fourth-order equations. Together with J. D. Tamarkin in [86] he studied the problem of the transverse vibrations of an elastic homogeneous rod fixed at the ends, that is an equation of the form

$$v_k^{(IV)}(x) = \lambda_k^4 v_k(x)$$

was considered for the homogeneous boundary conditions

$$v_k(0) = v_k'(0) = v_k(1) = v_k'(1) = 0$$

An asymptotic expression was found for an orthonormalised system of eigenfunctions $\{v_k(x)\}$ as $k \to \infty$, and a theorem was proved on the expansion of functions into Fourier series by the $\{v_k(x)\}$ system which satisfied supplementary conditions (a Lipschitz or a Dini condition or had a limited variation). An important result was obtained here, namely any function that could be expanded into a trigonometric Fourier series could also be expanded into the fundamental functions of the above equation. Finally, the Fejér theorem about the convergence to a continuous function of its average partial sums was proved in this work. Later, they generalised these results for the case of a nonhomogeneous rod.

All of Steklov's investigations that have been enumerated were related to the case where the fundamental functions possess orthogonality. An essentially new stage in the history of asymptotic methods in the theory of nonself-adjoint ordinary linear differential equations and the expansion of arbitrary functions into series was achieved by Tamarkin under the supervision of Steklov.

9

THE SCHOOL OF MATHEMATICAL PHYSICS

The Petersburg mathematical school was found by the brilliant mathematician P. L. Chebyshev (1821-1894), but its source goes as far back as L. Euler (1707-1783). It has played an outstanding role in the development of mathematics and mechanics both in Russia and abroad. For nearly all of the second half of the 19th century and the beginning of the 20th century, the themes of Chebyshev's work prevailed in Petersburg. The situation changed

somewhat when Lyapunov arrived from Kharkov in 1902 followed by Steklov in 1906 who soon established the first school of mathematical physics in Russia at Petersburg University. Note that a leading role in the development of mathematical physics had been played in Petersburg by M. V. Ostrogadskii (1801-1862) and by B. Ya. Bunyakovskii (1804-1889). Upon the appearance of their work the problems of the theory of partial differential equations became the centre of the attention of many of the representatives of the mathematical school in Petersburg (A. M. Lyapunov, V. A. Steklov, N. M. Gyunter, V. G. Imshenetskii, N. Ya. Sonin, et al.).

Up until the arrival of Steklov in Petersburg some noteworthy results had been obtained there in the theory of differential equations by such outstanding mathematicians as A. N. Korkin and A. A. Markov. The work of the Academician Lyapunov on mathematical physics (potential theory, stability theory, etc.) had a great impact on the creation and scientific activities of Steklov's school.

Steklov was not only an outstanding scientist who had enriched mathematics and mechanics by a series of important investigations, he was also a fine teacher and administrator. Under his continuous supervision scientists such as V. V. Bulygin (1888-1918), E. F. Gavrilov (1887-1961), M. F. Petelin (1886-1921), V. I. Smirnov (1887-1974), J. D. Tamarkin (1888-1945), A. A. Fridman (1888-1925) and Ya. A. Shokhat (1886-1944) have matured.

The activities of Steklov and the school he founded have been fruitful and many-sided. Characteristic of his work was the amalgamation of profound theoretical studies and the practical application of mathematical methods to the fundamentals of the natural science. A student of Kharkov University, the best pupil and successor to Lyapunov, Steklov considerably extended the themes of his teacher. Even whilst still at Kharkov University, Steklov belonged to the Petersburg mathematical school by the nature and direction of his own research. He started his research in areas related to mechanics (hydrodynamics and the theory of elasticity) but his most important results were obtained, as we have already noted, in mathematical physics. The majority of Steklov's work was related to boundary-value problems for partial and ordinary differential equations. He was the first to provide a rigorous justification of Fou-

rier's method of solving mixed problems of vibration equations for a nonhomogeneous string and mixed problems of cooling a nonhomogeneous rigid rod. Mathematically the question reduced to the Sturm-Liouville problem for an ordinary linear self-adjoint second-order differential equation with linear homogeneous boundary conditions. The basic problem was to prove the existence of an infinite set of eigenvalues and to solve the problem of expanding an arbitrary function into eigenfunctions. Steklov constructed asymptotic expressions for the eigenfunctions of the indicated problem. He obtained conditions for the expansion of an arbitrary function into the eigenfunctions that are as general as those for an expansion into an ordinary trigonometric Fourier series

The most important results Steklov achieved related to the solubility and representation of the solutions of Dirichlet's and Neumann's problems for Laplace's equation with the minimum number of assumptions for his time about the boundary of the domain and the boundary functions. His most outstanding success was the theory of completeness. His idea of averaging a function has led to the concept of functions of intervals, domains, and sets and has become a part of the apparatus of modern theory of partial differential equations. It has led to the concept of a generalised function and thus brought the apparatus closely in tune with the real conditions of physics experiments. Part of these results were obtained in the Kharkov period of his work.

The appearance of Steklov at Petersburg University had an immediate effect on its student and scientific life [XVI]. Within his first four years there, he had prepared several able students for scientific work. Three of these stayed on in 1910 at the faculty of mathematics for scientific and lecturing training [XLI].

A history of the founding of Steklov's school of mathematical physics and a description of its work in the first years of its existence is given in [XXIX].

A pupil of Steklov, Bulygin* was a very gifted and versatile mathematician. In the words of Prof. A. M. Zhu-

* V. V. Bulygin graduated from Petersburg University in 1910 and was taken in by Steklov into the faculty to work towards a scientific and lecturing career in the same year. He died when he was still young (in 1918) and never finished his dissertation.

rsvskii he was one of Steklov's favourite pupils. Bulygin atarted studying the theories of differential equations and elliptical functions. By the end of his short life he had discovered an exact formula using the theory of elliptical functions for the number of ways an integer N could be represented as a sum of r squares*. This result of Bulygin was a success of Russian mathematics.

The investigations of A. F. Gavrilov** were related to the equations of mathematical physics and numerical methods among others. He studied the non-linear generalisations of the telegraph equation and solved them by decomposing them into powers of a small parameter. He was applying to nonlinear partial differential equations, an idea Lyapunov and Krylov had developed for solving ordinary differential equations. He showed that the decomposition of the square of the frequency into the infinite series that is sufficient for destructing the secular terms when solving ordinary differential equations must be enlarged by the expansion of the other terms in the equation into power series in the case of partial differential equations of a small parameter. In this way he obtained the necessary parameters for destructing the secular terms.

Another pupil of Steklov, M. F. Petelin*** had shown in his first publication *On the Hydrodynamics Problem of Bjerknes***** written in association with A. A. Fridman, that he had great capacity and application. The work considered a law governing the changes in volume of two spheres interacting on each other at each moment of time according to the inverse square law.

One of the most outstanding of Steklov's pupils was V. I. Smirnov***** who later became an Academician.

* *Proc. Acad. Sc.* 1914, No. 6, pp. 389-404.

** A. F. Gavrilov graduated from Petersburg University in 1912 and was accepted into the faculty by Steklov in 1914 to work towards professorship. In 1919 he went to work at Nizhegorod University and from 1920 worked in various Institutes in Leningrad. He died on 30th September 1961 in Leningrad.

*** He graduated in 1911 from the mathematics department at Petersburg University and was accepted by Steklov into the faculty to do postgraduate work.

**** *Comm. Khar. Math. Soc.*, Ser. 2, V. XIII, No. 6, 1913, pp. 253-262.

***** He graduated from the mathematics department at Petersburg University in 1910. Steklov accepted him to work to-

He obtained (right at the start of his scientific career) a noteworthy result on the analytic theory of ordinary linear differential equations. He studied the following equation in his master's thesis:

$$[x(x-a)(x-1)y']' + (x+\lambda)y = 0$$

where $0 < a < 1$ and λ is a real parameter. Smirnov studied the problem in detail and in particular gave a complete picture of the spectrum of this equation. He also elucidated under which conditions the inversion problem would have a single-valued solution. Later he provided a qualitative (topological) criterion for the equivalence of two second-order linear equations with rational coefficients from the viewpoint of rational transformations of the independent variable. The results are applied to obtain equations with four singular points from Gauss's equations. In their review of this dissertation Academician Steklov and Professor Gyunter wrote: "...From all the above said, it is clear that this first major work of V. I. Smirnov (more than 400 pages) bears witness not only to the author's erudition and his capacity critically to examine new and sometimes quite complex though not too well-established theories, but also to his talent and original research. His work promises that in the future his activities will bring him a good scientific reputation..." [LXI]. Later Smirnov obtained important results in the theory of partial differential equations (functionally invariant solutions of the wave equation on a plane). He investigated (together with S. L. Sobolev) the seismic waves that are propagated from a point source located under the surface of the earth under the influence of an instantaneous shock arising at that point.

V. I. Smirnov penned the well-known *Course of Higher Mathematics*, a five-volume book that combined an exceptional wealth of material with a rigorous and masterly presentation.

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wards a professorship in 1912. In 1918 he defended his master's thesis, *The Problem of Inversing A Second-Order Linear Differential Equation with Four Singular Points*. In 1932 he became a Corresponding Member of the USSR Academy of Sciences and a Member in 1943. He died on the 14th February 1974 in Leningrad.

A pupil whose research was closely related to Steklov's was J. D. Tamarkin*. His early work was on the theory of ordinary linear differential equations of any order but with variable coefficients. He also studied systems of first-order equations which contain a large parameter λ. The coefficients of the equation or system have asymptotic expansions in the form of a series in negative powers of λ. He provided a method to integrate this sort of system using the series and derived asymptotic solutions for large $|\lambda|$. He later considered the boundary-value problem of quite a general form and explained the link between the main part of the corresponding Green function, the eigenfunction and adjoint function. He studied the transcendental equation for characteristic values and the expansion of Green's functions into simplest fractions.

Finally he studied the expansion of an arbitrary function into the eigenfunctions and adjoint functions of the boundary-value problem. He was the first to consider the case of multiple roots of the corresponding characteristic equation and to construct asymptotic expressions containing fractional powers of the parameter λ.

Tamarkin's work which appeared soon after that of Birkhoff, was the first in Russia on non-selfconjugate boundary-value problems for ordinary linear differential equations.

Steklov wrote in his review of Tamarkin's master's thesis "... This author's work is a development and generalisation of results both he and other authors have already obtained... In subsequent chapters he establishes a general method for finding the asymptotic expressions to solve differential equations. The method includes Schlesinger's and Birkhoff's results as special cases. He found

* He graduated in 1910 from Petersburg University's mathematical department and was accepted by Steklov to do postgraduate study. In 1917 he defended his master's dissertation *On Some General Problems in the Theory of Ordinary Linear Differential Equations and on the Expansion of Arbitrary Functions into Series.* He later taught mathematics in Petersburg and Perm. In 1925 he emigrated to Latvia and then to America where he died in 1945. There is an article by E. Hille about his life and work in *Bul. Amer. Math. Soc.*, 1947, Vol. 53, No. 5, pp. 440-457, in which there is a bibliography of his work.

the solution to problem (B)* on Cauchy's method, and gave a number of new propositions about the so-called Green function, for example, when expanding these functions into simplest fractions. Tamarkin used his results to solve, finally, the problem of expanding of an arbitrary function (problem (B) generalised) for some very general assumptions. Moreover he did not limit himself to orthogonal solutions and obtained an expansion of a new type. This included functions other than orthogonal ones and the thesis's author called them principal functions..."** [XLI, No. 10375, pp. 57-59].

At the same time as he was investigating the boundary-value theory for differential equations. Tamarkin was occupied with other questions related to various branches of mathematical analysis and mathematical physics. In 1915 he published a work in which he studied the problem of going to the limits of integrands. Starting in 1918 he published several works on approximation methods for solving differential equations. Later work was related to various topics of the theory of functions of real and complex variables.

Another of Steklov's pupils was A. A. Fridman***. Fridman was already studying partial differential equations of the elliptic type as an undergraduate. Later he became well-known among those who studied mechanics, dynamic meteorology and the theory of relativity. In October 1925 in Fridman's obituary, Steklov wrote, "The late director of the Central Geophysical Observatory, Aleksandr Aleksandrovich Fridman, was one of the best of my pupils, and at the proper time he was retained by me at Petersburg University to study towards a professorship in applied mathematics. Even as an undergraduate he had published several investigations which all proved his wide knowledge of mathematics and his ability for inde-

* Problem (A) is to find a solution of an ordinary linear differential equation for certain conditions at the ends of a segment and problem (B) is the expansion of an arbitrary function into series in the above solutions.

** Principal functions are now called adjoint functions.

*** He graduated from Petersburg University's mathematics department in 1910 and was accepted by Steklov and D. K. Bobylev for postgraduate work. In 1922 he defended his dissertation *Experience of the Hydrodynamics of Compressible Liquids.*

pendent work. A brilliant mathematician who not only did work on analysis but even applied his own methods to the study of the basic problems of fluid dynamics, particularly to the dynamics of the atmosphere which must in consequence underlie scientific meteorology. A. A. Fridman was one of the pioneers of theoretical meteorology to the study of which he devoted his strength during the last years of his life. His investigations of various practical topics in the dynamics of the atmosphere have gained a wide reputation and contain some new and valuable results. The extraordinary breadth of his talent and his rare capacity for work should be noted. The World War pulled him away from his purely scientific work, but even in the most unfavourable conditions, as an airforce pilot, during a Russian offensive on the North-West front and during the capture of Peremyshl, he still found time for scientific work ..." [153].

The work of J. A. Shohat* was devoted to the study of the theory of polynomials which least deviated from zero in finite and infinite intervals. He also studied the problem of moments among other topics of mathematics. In his dissertation he studied polynomials of degree n (whose coefficients were linearly related) which deviated the least from zero in a given interval (finite or infinite).

When Steklov arrived in Petersburg and became one of those responsible for the mathematics training at the University, N. M. Gyunter (1871-1941) had gained considerable experience of teaching mathematics in higher educational establishments. He became Steklov's active helper in the organisation of the school of mathematical physics.

Of all the Petersburg mathematicians, Gyunter was the closest to Steklov as far as their research was concerned. Later his scientific work began to show the strong influence of Steklov's scientific achievements. Gyunter's basic ideas in mathematical physics had their origins in Stek-

* J. A. Shohat graduated in 1910 from Petersburg University and was accepted by Steklov in 1911 as a postgraduate student. In 1922 he defended his thesis *Investigation of One Class of Polynomials which Least Deviate from Zero in a Given Interval.* The same year he obtained permission from the Soviet government to be repatriated to Poland (his parents were of Polish extraction) whence he soon emigrated to America, dying there in 1944.

lov's work on smoothing functions, and a large series of his works were related to the application of the method of smoothing functions. The method led to the discovery of more general formulations of the boundary-value problems of mathematical physics. Gyunter systematically applied the idea of a function of a domain (set) and Stieltjes's integral to the investigation and solution of these problems and expounded potential theory in these terms.

Steklov also had a considerable effect on the work of other outstanding Leningrad mathematicians such as I. M. Vinogradov, A. M. Zhuravskii, N. S. Koshlyakov, N. E. Kochin, R. O. Kuzmin, I. A. Lappo-Danilevskii, and N. G. Chetaev.

Steklov's work came at a turning point of the history of mathematical physics. By refining older methods and creating newer ones, he started down a completely new path of mathematical research and anticipated the fruitful ideas of modern mathematics, the mathematics of the second half of the twentieth century. These are primarily related to his theory of completeness and the method of smoothing functions.

After the death of V. A. Steklov, the traditions of the Leningrad school of mathematical physics were successfully maintained by N. M. Gyunter, I. A. Lappo-Danilevskii, A. N. Krylov, R. O. Kuzmin, V. I. Smirnov, S. L. Sobolev, V. A. Fok, N. P. Erugin, and others.

10

V. A. STEKLOV—HISTORIAN OF MATHEMATICS, PHILOSOPHER AND WRITER

V. A. Steklov was not just an outstanding mathematician, but a wonderful teacher and administrator of scientific investigations. From the days of his youth he frequently pondered the role mathematics played in the material and cultural development of a society. He viewed the development of mathematics from the standpoint of materialist philosophy and his ideas here can be seen to have been influenced by the revolutionary democrats I. A. Dobrolyubov, N. G. Chernyshevskii, D. I. Pisarev and others.

From the start of the February Revolution Steklov and Gorky began actively to popularise science. It was from

this time that he began to work on those of his notes that are on the history of mathematics and philosophy.

Steklov was always trying to bring science closer to the people and to strengthen its influence on the life of the society. He did not have just the application of science in mind for he said, "Science is the unequalled educator of man."

Steklov was a master of the difficult genre of creative scientific prose. He wrote two books that were biographical in nature, one on M. V. Lomonosov [147] and the other on Galileo Galilei [148]. Steklov believed Lomonosov was one of the greatest figures in Russian science and in his opinion Lomonosov was not appreciated by his contemporaries. He wrote, "Only one person in Lomonosov's time clearly understood the profundity and originality of his genius and that was Euler, one of the greatest geometers in the world. This we know only through Euler's personal letters... The majority of scientists who were Lomonosov's contemporaries treated his ideas with indifference and sometimes even with animosity..." [147, p. 6].

It should be noted that until the 1920's Lomonosov was only known as a writer. The views that we now have of him as a genius of Russian science are due, among others, to Steklov's book. Steklov conjured up a living portrait of this great and original scientist who rose out of his unlettered peasant background in the savage North to open up such wide horizons in chemistry, physics, and the other sciences that they could not be embraced either by many of his contemporaries in Russia or by a majority of scientists in the West. Lomonosov's activities, particularly his scientific achievements, were described by Steklov in the deep and interesting way that only a well-rounded scientist himself accredited with large contributions to science and a master of the creative word could.

Steklov wrote an excellent biographical essay of Galileo Galilei [148] in which he not only made a detailed survey of the scientist's life but also clearly described the science of his times and the new avenues Galileo opened up. Steklov set forth the heroic struggle Galileo had with the ruling classes in a time of medieval prejudice and religious fanaticism.

A brilliant example of Steklov's popularising activities was a speech he gave on 16th May 1921 at a celebration

given by the Academy of Sciences to mark the 100th anniversary of the birth of P. L. Chebyshev [145]. With unusual clarity he drew a portrait of the man who had combined deep theoretical investigations with genuinely practical problems.

Steklov also wrote about the life and activities of N. I. Lobachevskii [XXXII, op. I, No. 130, pp. 1-15], M. V. Ostrogradskii [131] and other scientists.

In 1920 Steklov finished work on a book with a historical and philosophical theme *Mathematics and Its Significance for Man* [149]. Writing about this book of his he penned, "I wanted on one hand to establish from a short historical perspective the intimate connection between mathematics and all the systems of philosophy beginning with the most ancient. I also wanted to show that mathematics always has been and always will be the source of philosophy, that it created philosophy and so can perhaps be called the 'mother' of philosophy.

"On the other hand I subsequently tried to retrace in a general term the movements of philosophical thought about the origins and certainty of mankind's knowledge... and in particular the origins and character of the basic postulates of geometry" [149, pp. 30-31]. Steklov wrote this short historical survey of the development of mathematics and its influence on philosophy in the bright, accurate, and laconic language of mathematics. He concluded that mathematics was the source of philosophy.

In setting out his own views, Steklov affirmed that everything in nature and society must one day become an object of mathematics. Mathematics arose and developed from experience, as a result of the practical activities of people. He rejected Kant's view that mathematics is an *a priori* science that can come into being purely by thought. He was against Kant's agnosticism and his assertion that it was impossible to comprehend "a thing itself". He wrote, "...I recognise that the ability to perceive certain external impressions in the form of spatial sensation, certain colours, etc., is a property of our nature and was incorporated in us before any experience was gained. Still I believe that this ability cannot come of its own accord, it cannot begin to work before we have obtained our first sensation from the outside, that is the ability to perceive

could not come before the simplest form of experience, since man, by his very nature, is born with the potential ability of perceiving a certain group of sensations in the form of, for instance, the spatial sensations and nothing else, and another group in the form of colours, etc. The ability to apprehend sensations, to sense their similarities and differences (i.e. compare them one to another), to recognise them again in the mind is the nature of the organ we now call man's brain..." [149, p. 39].

Steklov considered that the bases of all sciences, including pure mathematics (which according to him embraces arithmetic, algebra and partially geometry), were set down by a long line of experiments, observations, and generalisations of what was known from the many special cases of the general laws. He assigned a special role to intuition as one of the forms of cognition. He wrote, "... It seems to us that the special property of the mind of man that is displaced here and that is the heart of his creativity and is one of the tools of discovery and invention, the one we now call intuition. This word though is not always given the same meaning we are now talking about..." [149, p. 104]. Later he wrote, "The method of discovery and invention is one and the same for everyone, intuition, for none would discover anything just using logic. Syllogism can only lead others to the recognition of some previously known verity, but as a tool for invention it is useless... A mathematician sometimes suggests a rather complicated proposition which is not at all clear and then begins to prove it... In discovering almost every step in the proof it is intuition, rather than logic, that plays the main role and prevails over all the logic" [149, p. 110].

As he was criticising various philosophical systems Steklov came to the conclusion that, "Innate or *a priori* ideas do not exist in the mind of man. All the axioms and laws of the sciences of nature, starting with mathematics, arise in his mind from experiments and observation. However, the ability to extract the axioms and laws out of the experience accumulated in the mind by the same ability mentioned above (i.e. intuition). It is the really innate property of the mechanism we call the mind. The presence of the intuitive capacity can be established by direct observation..." [149, p. 111].

According to Steklov we cannot know anything for

certain because our minds are imperfect and can conceal the truth. In this respect Steklov adhered to the same point of view as Lobachevskii, "Stop struggling in vain to arrive at all knowledge through the mind. Ask Nature for she guards every secret and will answer your every question accurately and satisfactorily." It is quite understandable that Steklov held the same views as Lobachevskii and the rationalists and the idealistic school of Kant were dealt a severe blow by Lobachevskii when he created non-Euclidean geometry.

Steklov concluded his book with praise of the contributions of great Russian mathematicians, "This trend in the theory of the cognition of external world was created by the geniuses of mathematical thought. Of these one of the first places belongs to Russian mathematicians and of these the first would be Lobachevskii and Chebyshev jointly" [149, p. 137].

As regards Steklov's views of the significance of mathematics, Lunacharskii wrote, "He was a persuasive supporter of the purely empirical emergence of mathematics and greatly disapproved of the idealists and formalists in the science. He was continually reiterating that mathematics was earthly but he believed that the mathematical formulation of natural phenomena was the limiting lucidity of a truth. He once said to me that people will all agree on every question only when a science of nature, i.e. all verity, is mathematically formulated. He laughed triumphantly and cast a glance at me slyly, he stroked his prophet-like beard and continued, 'You can't argue with mathematics...' I would like to draw the reader's special attention to the popular brochure written by Steklov about mathematics. Everyone trying to attain a general education can read it with benefit" [XXXIV].

Steklov's great ability for creative writing is shown up in *To America and Back. Impressions* [151] and in the narrative about his life [XXXII]. In [151], written about his visit to the International Mathematical Congress in Canada, he described everything in an interesting way. He covered the American's tenor of life, the customs and tastes of their various classes and nationalities of America, the splendour of the broad ways in Western cities, and the squalor of the outlying areas. Steklov introduced his readers to the large hall of the meeting of the representatives

of world science and to the close atmosphere of exchange. He described nature both simply and beautifully. In the same book he described the ceremony in which he was awarded the honorary degree of doctor of Toronto University for his outstanding work in mathematics and mechanics.

The huge responsibility of Steklov before his own people is staggering. Whilst abroad he turned out to be interested in a host of things which seemed to be extraneous, such as new machinery, factories, and mines. On arriving home he compiled detailed notes with recommendations as to how to use the new technology in the economy of Russia.

Steklov's work on history and the philosophical methodology of mathematics, his popularisations of science, and his brilliant articles and speeches on Chebyshev [145], Lyapunov [144], Markov [146], Poincaré [142], and Thomson [150] among others have lost none of their relevance for today.

IN PLACE OF A CONCLUSION, A SUMMARY

The legacy and deeds of a scientist are subject to the test of time. They will be kept and developed only if they continue to serve progress and people's practical activities. The work of V. A. Steklov has indubitably endured that test. It has had and continues to have an influence on the development of mathematics and mechanics. Academician Krylov in 1936 said of Steklov, "...he shoud be numbered with that group of significant Russian mathematicians that includes Ostrogradskii, Chebyshev, and Lyapunov" [XIX].

The scientific interests of Steklov, as we have seen, embraced an extraordinary wide range of topics. These include fluid dynamics, the theory of elasticity, analytical mechanics, geophysics, the history of science and philosophy. His particular interest covered many of the topics of mathematical physics and analysis such as the boundary-value problem for the Laplace equation, the theory of fundamental functions, justification of Fourier's method, the theory of completeness, asymptotic methods, expansion of functions into series, quadrature formulae, approximation of functions, and orthogonal polynomials.

Steklov's basic work was in mathematical physics and he achieved some outstanding success in the field. By refining older methods and creating newer ones he started down a completely new path of mathematics research and anticipated the fruitful ideas of modern mathematics. These are primarily related to its theory of completeness, the method of smoothing (averaging) functions, and the functional inequalities like the imbedding theorem. He endowed many branches of mathematics with the rigour and exactness that is a characteristic of the Petersburg mathematics school. In the breadth of his work and the profound usefulness of his methods, Steklov is a model in the field of mathematics

Steklov was an outstanding educator and administrator of science and rising generation of scientists. He was an excellent writer, populariser of scientific knowledge, and a philosopher who deeply understood culture.

The educational activities and the work Steklov did for society at large were highly appreciated by his contemporaries. He was elected a Corresponding Member of the Academy of Sciences in 1902, made an adjunct Member in 1910 and became a full Member in 1912. From 1919 until the end of his life, Steklov was the Vice-President of the Academy of Sciences. He was a member of many mathematical societies such as the ones in Kharkov, Moscow, Petersburg/Leningrad, and Palermo; he was a doctor honoris causa of the University of Toronto; a Corresponding Member of the Academy of Sciences in Goettingen; a member of the German Seismological Society in Jena; and so on.

His greatest merit was that he was the first person in the country to give Petersburg University its own school of mathematical physics. Many authoritative scientists matured under his unwavering supervision.

During some very difficult years in the country—the time of the Civil War and economic dislocation—Steklov guided the Academy of Sciences, both in its scientific work and in its internal affairs. He set right the printing of scientific works and obtained scientific literature and equipment from abroad. He worked a great deal on the reestablishment of the ruined seismological network. He was a member of many commissions, such as the Commission to Study the Productive Forces under Gosplan, the

Committee for Science under Sovnarkom, and was the Chairman of the Standing Commission for Seismology, and so on. Steklov would not just participate but would always be a figure of energy and initiative.

Steklov's role in the organisation of the Physico-mathematical Institute of the Academy of Sciences should be emphasised. He was its director from inception (1921) until the end of his life. In 1934 the Institute was divided into two: the P. N. Lebedev Institute of Physics and the V. A. Steklov Mathematical Institute. In fact the start of the Mathematical Institute can be dated to 28th November 1932 when Academician I. M. Vinogradov was elected its director*. Today the Mathematical Institute of the USSR Academy of Sciences is world renowned and carries the name Vladimir Andreevich Steklov by right and with pride.

Steklov's name will remain for ever in science. He will be remembered not only as a significant scientist but as one of the outstanding organisers of Soviet Science and brilliant educator.

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* Even earlier on 28th February 1932, the general assembly of the USSR Academy of Sciences had decided to split the Physico-mathematical Institute.

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Abbreviations of the Periodicals in Which the Work was Published

СХМО	— Сообщения Харьковского математического общества (Communications of the Kharkov Mathematical Society).
Math. Ann.	— Mathematische Annalen.
ТОФНОЛЕ	— Труды Отделения физических наук общества любителей естествознания (Proceedings of the Physics Department of the Society of the Lovers of Science).
CR	— Comptes rendus des séances de l'Académie des sciences de Paris.
ЗХУ	— Записки Харьковского университета (Transactions of Kharkov University).
AFT	— Annales de la Faculté des Sciences de Toulouse.
JRAM	— Journal für die reine und angewandte Mathematik.
B.l'Ac.Crac.	— Bulletin de l'Académie des sciences de Cracovie.
ЗФМО	— Записки Императорской Академии наук по физико-математическому отделению (Transactions of the Physico-Mathematical Section of the Imperial Academy of Sciences).
AEN	— Annales scientif. de l'Ecole normale supérieure.
RAL	— Rediconti della Reale Accademia del Lincei.
RCMP	— Rendiconti del Circolo Matemático de Palermo.
ИАН (IAN)	— Известия Академии Наук (Izv. Akad. Nauk) (Proceedings of the Academy of Sciences).
ДАН (DAN)	— Доклады Российской Академии наук (Reports of the Russian Academy of Sciences).

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